Combinatorial Solving with Provably Correct Results

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Combinatorial Solving and Optimisation

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21][1]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP-complete problems (or worse) very successfully in practice!
- Except solvers are sometimes wrong... (Even best commercial ones) [BLB10, CKSW13, AGJ⁺18, GSD19, GS19, BMN22, BBN⁺23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

[1] See end of slides for all references with bibliographic details

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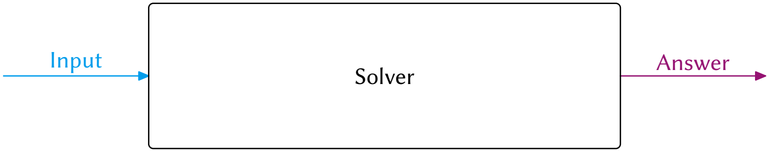
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by outputting
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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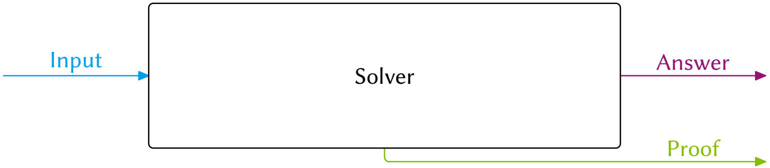

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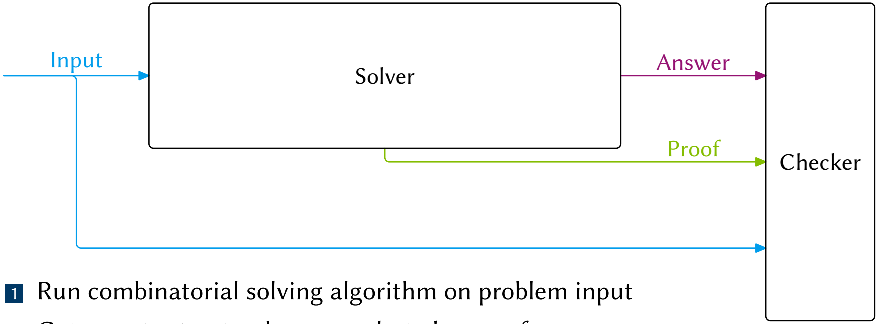

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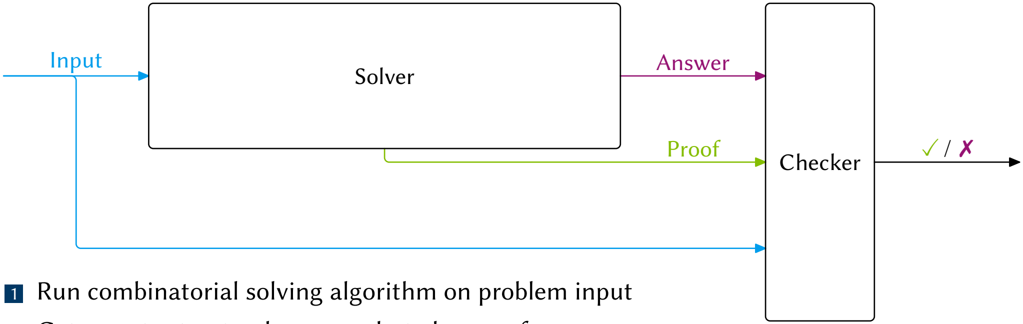

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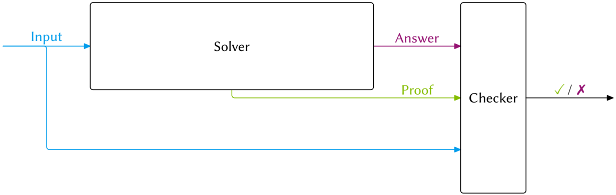
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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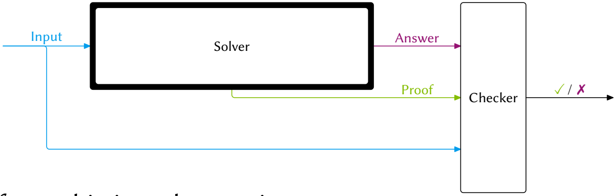

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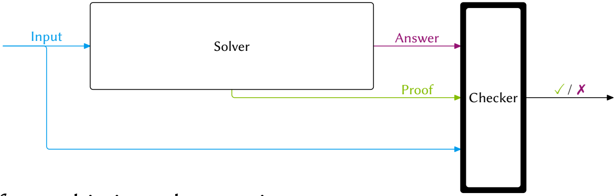

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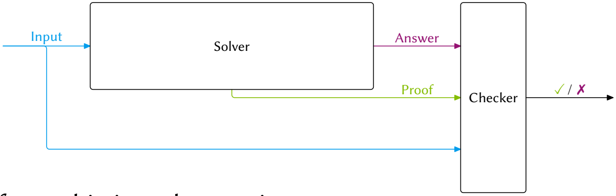

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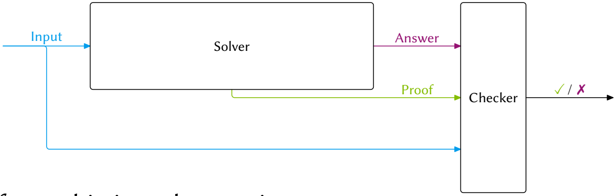

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Asking for both perhaps a little bit too good to be true?

Take-Away Message from This Tutorial

Proof logging for combinatorial optimisation is possible with single, unified method!

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Proof logging for combinatorial optimisation is possible with single, unified method!

- Build on successes in proof logging for SAT solvers with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during development [EG21, GMM+20, KM21, BBN+23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

The Rest of This Tutorial

Explain how to use VeriPB to do proof logging for

- SAT solving (including advanced techniques)
- SAT-based optimisation (MaxSAT)
- Subgraph algorithms
- Constraint programming

in a unified way

The SAT Problem

- Variable x: takes value **true** (=1) or **false** (=0)
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \lor \cdots \lor \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land$$
$$(x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

Proofs for SAT

For satisfiable instances: just specify satisfying assignment

For unsatisfiability: a sequence of clauses (CNF constraints)

- Each clause follows "obviously" from everything we know so far
- Final clause is empty, meaning contradiction (written $\perp$)
- Means original formula must be inconsistent

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Unit Propagation

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Proof checker should know how to unit propagate until saturation

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DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

"Proof trace": when backtracking, write negation of guesses made

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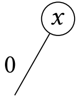

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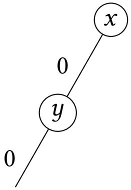

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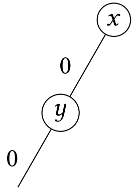

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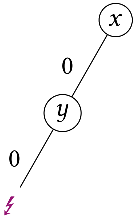

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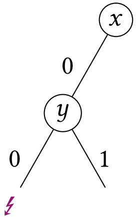

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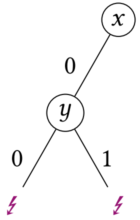

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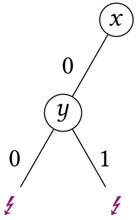

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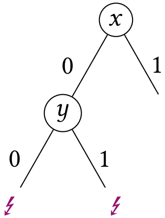

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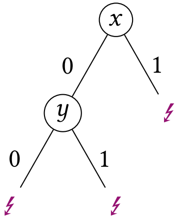

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1. $x \lor y$
2. $x \lor \overline{y}$
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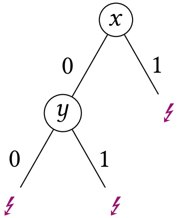
4. $\overline{x}$
5. $\perp$



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Fact

Backtrack clauses from DPLL solver generate a RUP proof

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Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

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Notation $p \stackrel{\mathrm{d}}{=} 0$

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Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathrm{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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$p \overset{\mathrm{d}}{=} 0$

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Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

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$p \overset{\mathrm{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

$q \overset{\mathrm{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathrm{d}}{=} 0$

Unit propagation

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$p \overset{\mathrm{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

$q \overset{\mathrm{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

Decision

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$p \stackrel{\mathrm{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathrm{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathrm{d}}{=} 0$

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$p \overset{\mathrm{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

$q \overset{\mathrm{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

$x \overset{\mathrm{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathrm{d}}{=} 0$

Unit propagation

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Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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$p \stackrel{\mathrm{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathrm{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathrm{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathrm{d}}{=} 0$

Unit propagation

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$p \overset{\mathrm{d}}{=} 0$

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$q \overset{\mathrm{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

$x \overset{\mathrm{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

$\overset{\overline{y} \lor \overline{z}}{\bot}$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

- $p \overset{\mathrm{d}}{=} 0$
- $u \overset{p \vee \overline{u}}{=} 0$

decision level 1

- $q \overset{\mathrm{d}}{=} 0$
- $r \overset{q \vee r}{=} 1$
- $w \overset{\overline{r} \vee w}{=} 1$

decision level 2

- $x \overset{\mathrm{d}}{=} 0$
- $y \overset{u \vee x \vee y}{=} 1$
- $z \overset{x \vee \overline{y} \vee z}{=} 1$
- $\overset{\overline{y} \vee \overline{z}}{\bot}$

decision level 3

Decision

Free choice to assign value to variable

Notation $p \overset{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \overset{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

decision level 1:
- $p \overset{\mathrm{d}}{=} 0$
- $u \overset{p \vee \overline{u}}{=} 0$

decision level 2:
- $q \overset{\mathrm{d}}{=} 0$
- $r \overset{q \vee r}{=} 1$
- $w \overset{\overline{r} \vee w}{=} 1$

decision level 3:
- $x \overset{\mathrm{d}}{=} 0$
- $y \overset{u \vee x \vee y}{=} 1$
- $z \overset{x \vee \overline{y} \vee z}{=} 1$
- $\overset{\overline{y} \vee \overline{z}}{\bot}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \overset{\mathrm{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

decision level 1

$q \overset{\mathrm{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

decision level 2

$x \overset{\mathrm{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

$\overset{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Could backtrack by erasing conflict level & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Trail	
$p \stackrel{\mathrm{d}}{=} 0$	decision level 1
$u \stackrel{p \vee \overline{u}}{=} 0$	
$q \stackrel{\mathrm{d}}{=} 0$	decision level 2
$r \stackrel{q \vee r}{=} 1$	
$w \stackrel{\overline{r} \vee w}{=} 1$	
$x \stackrel{\mathrm{d}}{=} 0$	decision level 3
$y \stackrel{u \vee x \vee y}{=} 1$	
$z \stackrel{x \vee \overline{y} \vee z}{=} 1$	
$\stackrel{\overline{y} \vee \overline{z}}{\bot}$	

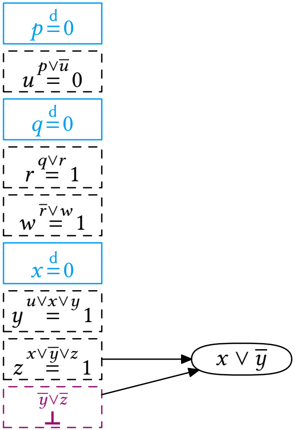
Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$



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But want to learn from conflict and cut away as much of search space as possible

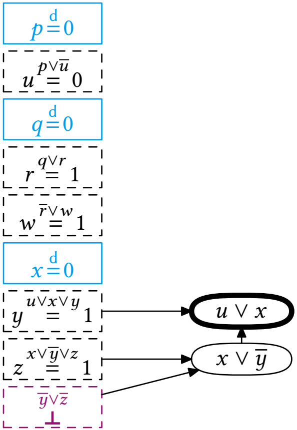
Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \lor \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$



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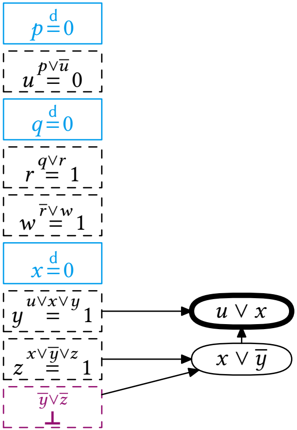
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \vee \overline{y}$

Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

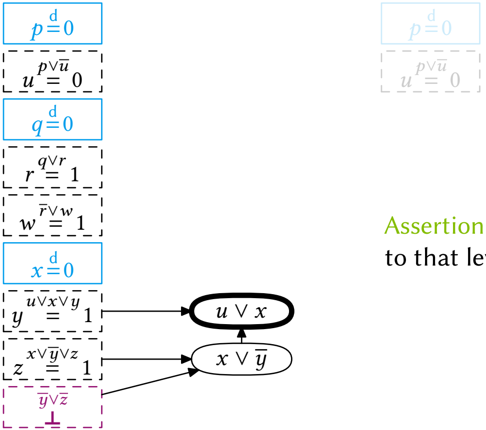
$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$

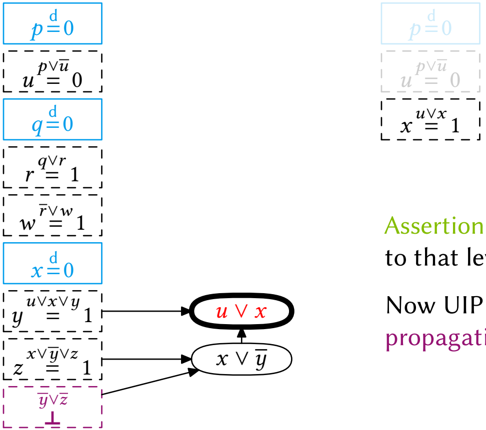


Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$



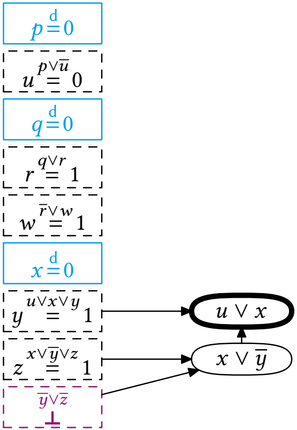
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

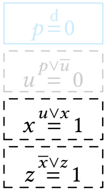
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$





Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

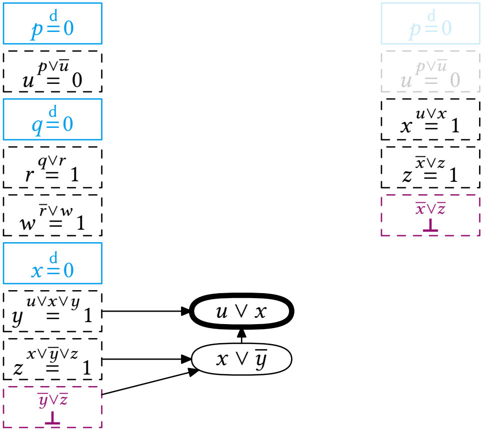
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Then continue as before...

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

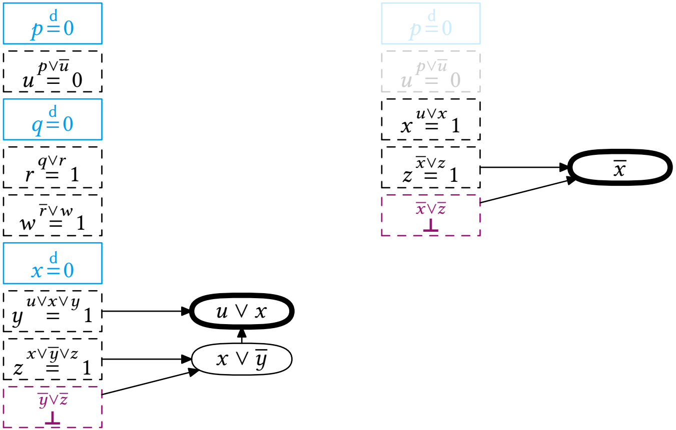
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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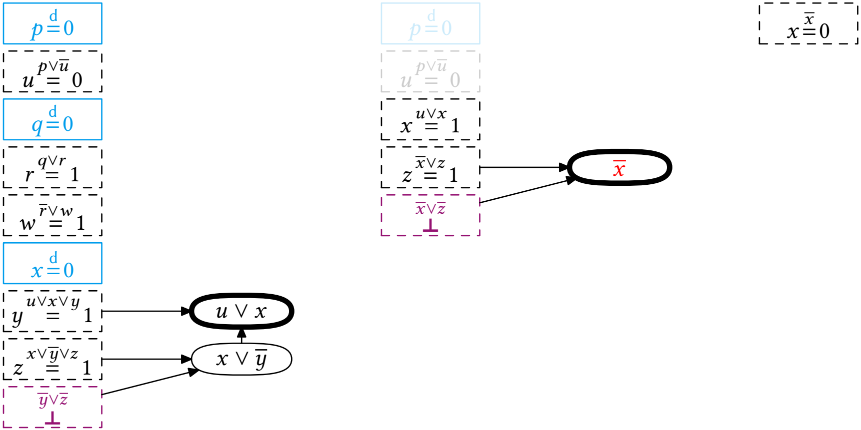
Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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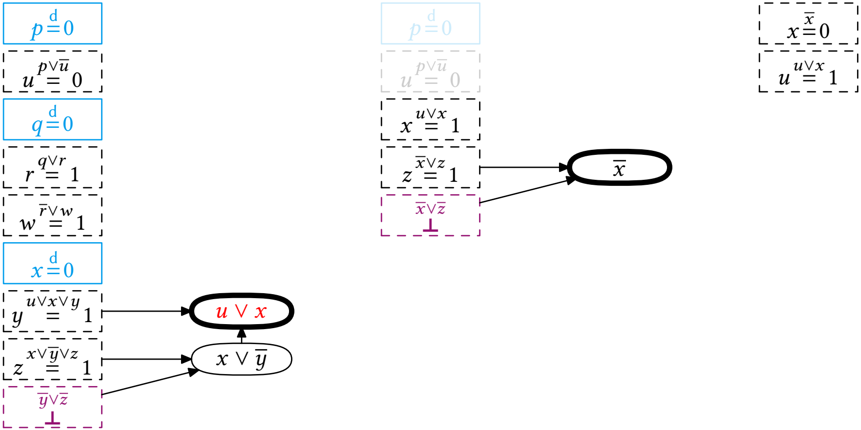

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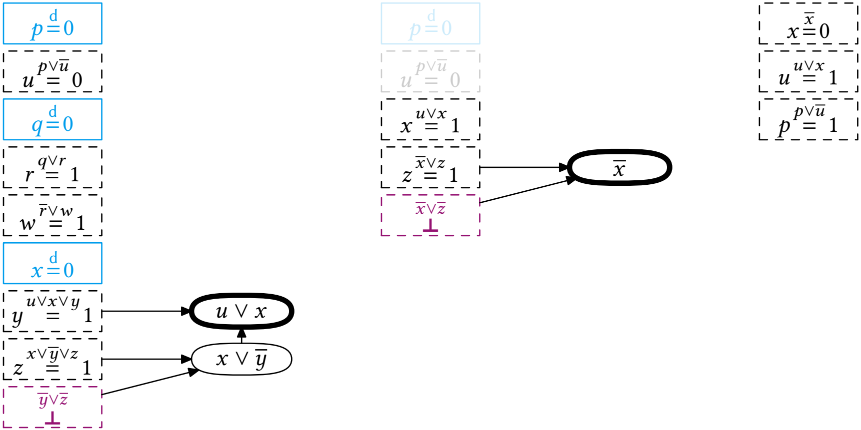
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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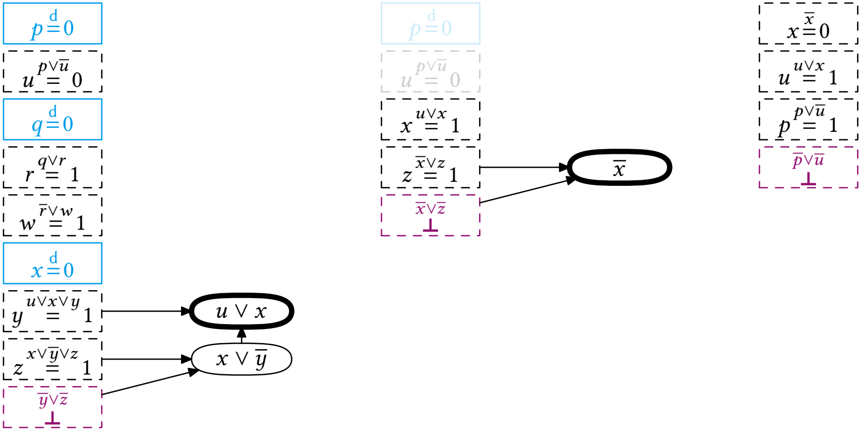
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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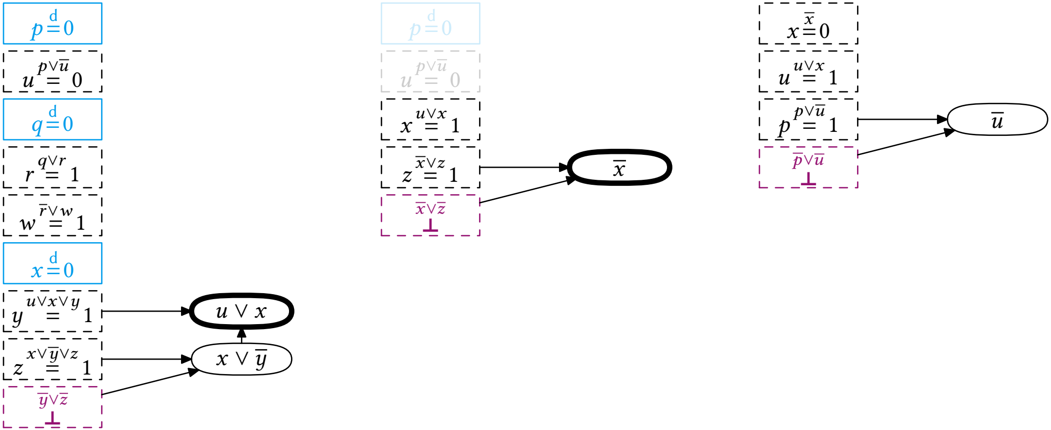
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

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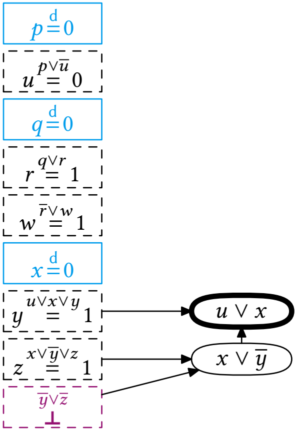
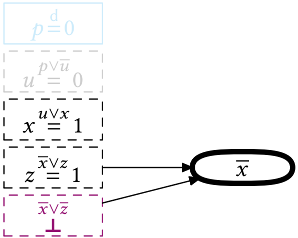
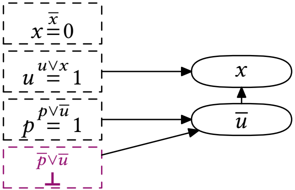
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



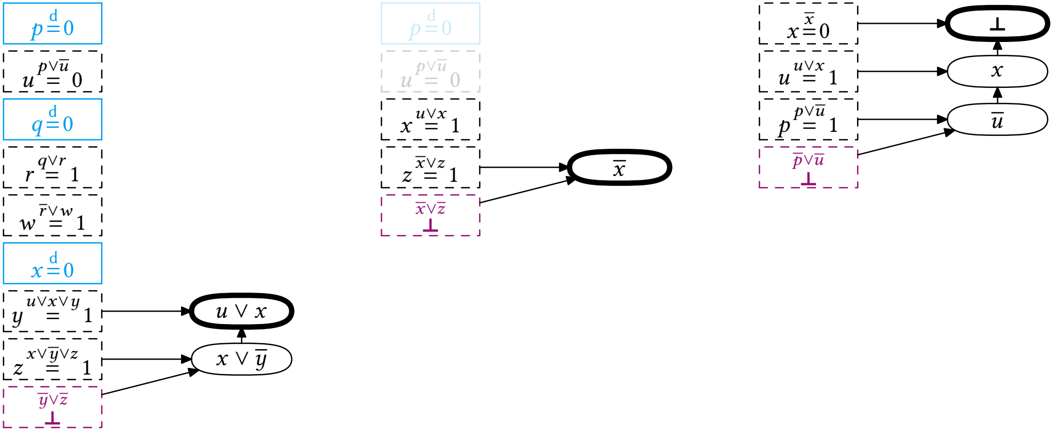




Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$



CDCL Reasoning and the Resolution Proof System

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Resolution proof system [Bla37, Rob65]

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$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

- Done when contradiction $\perp$ in form of empty clause derived

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(*) Ignores pre- and inprocessing, but we will get there...

Resolution Proofs from CDCL Executions

Obtain resolution proof...

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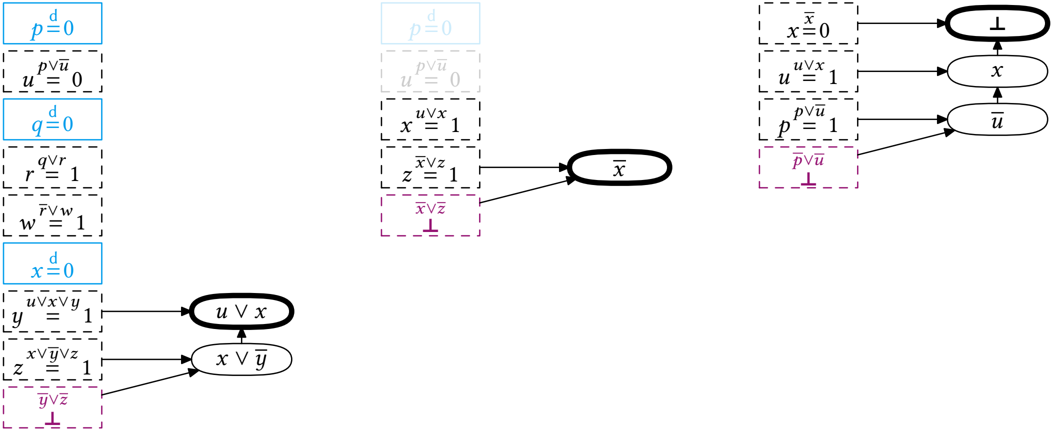
Obtain resolution proof from our example CDCL execution…



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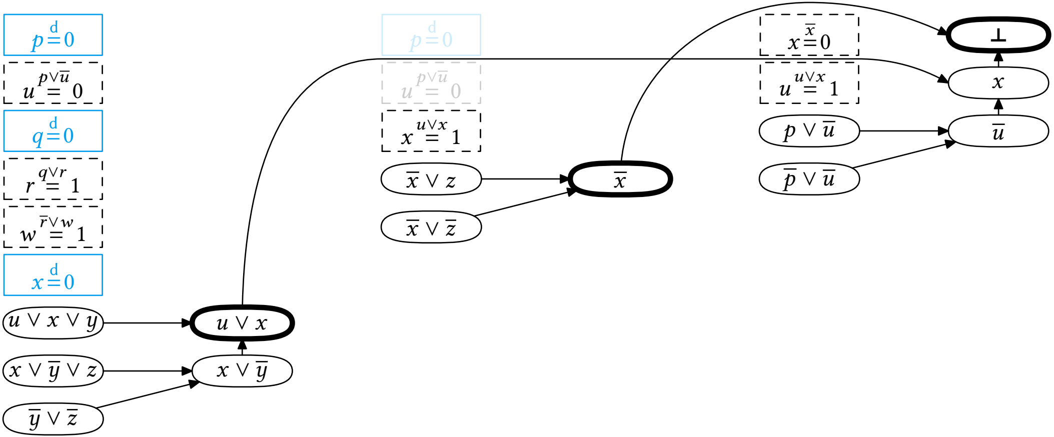
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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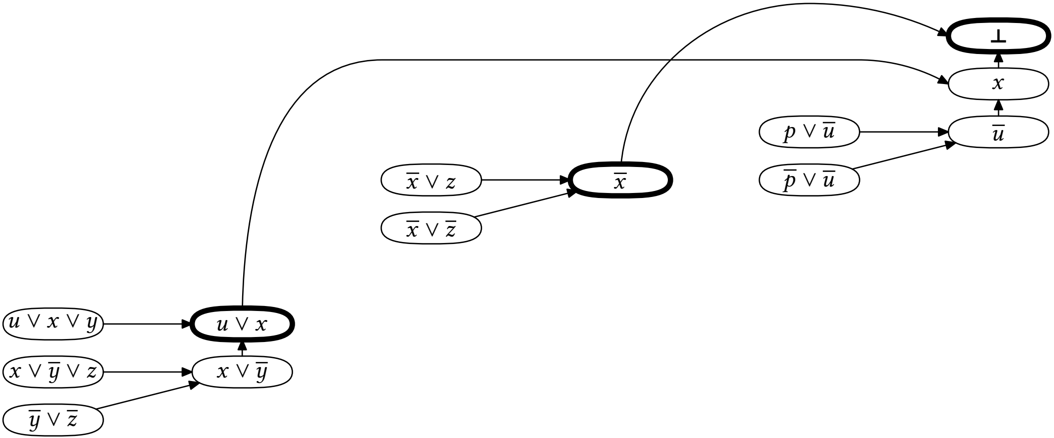

RUP Proofs and CDCL

But it turns out we can be lazier. . .

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All learned clauses generated by CDCL solver are RUP clauses

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$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

is sequence of reverse unit propagation (RUP) clauses

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2. $\overline{x}$
3. $\perp$

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More Ingredients in Proof Logging for SAT

Fact

RUP proofs can be viewed as shorthand for resolution proofs

See [BN21] for more on this and connections to SAT solving

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of reasoning

Extension Variables, Part 1

Suppose we want a variable a encoding

$$a \Leftrightarrow (x \land y)$$

Extended resolution [Tse68]

Resolution rule plus extension rule introducing clauses

$$a \lor \overline{x} \lor \overline{y} \qquad \overline{a} \lor x \qquad \overline{a} \lor y$$

for fresh variable a (this is fine since a doesn't appear anywhere previously)

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$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

for fresh variable a (this is fine since a doesn't appear anywhere previously)

Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system most commonly used for SAT solving

Why Aren't We Done?

Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
- Clausal proofs can't easily reflect what algorithms for other problems do

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Surprising claim: a slight change to 0-1 integer linear inequalities does the job!

- Enables proof logging for advanced SAT techniques so far beyond reach for efficient DRAT proof logging:
  - Cardinality reasoning
  - Gaussian elimination
  - Symmetry breaking
- Supports use of SAT solvers for optimisation problems (MaxSAT)
- Can justify graph reasoning without knowing what a graph is
- Can justify constraint programming inference without knowing what an integer variable is

Pseudo-Boolean Constraints

0–1 integer linear inequalities or (linear) pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)

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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x_1 \lor \overline{x}_2 \lor x_3 \quad \Leftrightarrow \quad x_1 + \overline{x}_2 + x_3 \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

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Literal axioms $$\frac{}{\ell_i \geq 0}$$

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Literal axioms $\dfrac{}{\ell_i \geq 0}$

Addition $\dfrac{\sum_i a_i\ell_i \geq A \qquad \sum_i b_i\ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$
(assumes normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

$$\text{Multiply by 2} \quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Cutting Planes Toy Example

Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

$w + 2x + 4y + 2z \geq 5$

Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Multiply by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

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}\ \text{Add}
$$

Naming constraints by integers and literal axioms by the literal involved (with ~ for negation) as

$$
\begin{aligned}
\text{Constraint 1} &\doteq 2x + y + w \geq 2 \\
\text{Constraint 2} &\doteq 2x + 4y + 2z + w \geq 5 \\
\sim z &\doteq \overline{z} \geq 0
\end{aligned}
$$

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$$\text{Constraint 2} \doteq 2x + 4y + 2z + w \geq 5$$
$$\sim z \doteq \overline{z} \geq 0$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d
```

Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\overline{y} \vee \overline{z} \qquad x \vee \overline{y} \vee z}{x \vee \overline{y}}$$

we can perform the cutting planes steps

$$\text{Add}\ \frac{\text{Divide by 2}\ \dfrac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{x + 2\overline{y} \geq 1}}{x + \overline{y} \geq 1}$$

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$$\frac{\overline{y} \lor \overline{z} \qquad x \lor \overline{y} \lor z}{x \lor \overline{y}}$$

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Given that the premises are clauses 7 and 5 in our example CNF formula, using references

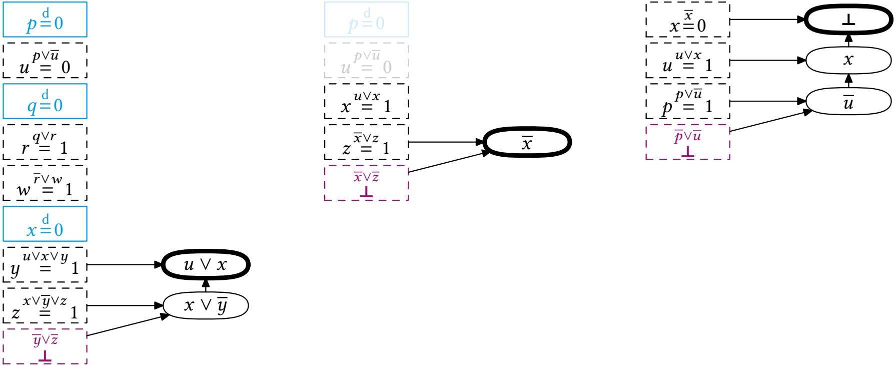
$$\text{Constraint 7} \doteq \overline{y} + \overline{z} \geq 1$$

$$\text{Constraint 5} \doteq x + \overline{y} + z \geq 1$$

we can write this in the proof log as

```
pol 7 5 + 2 d
```

Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



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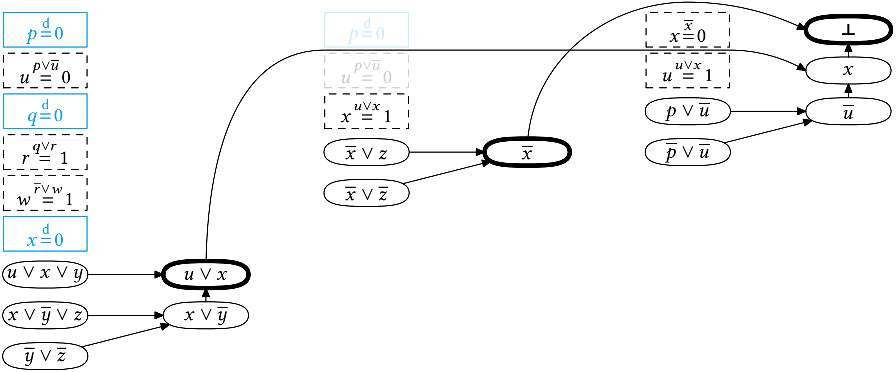

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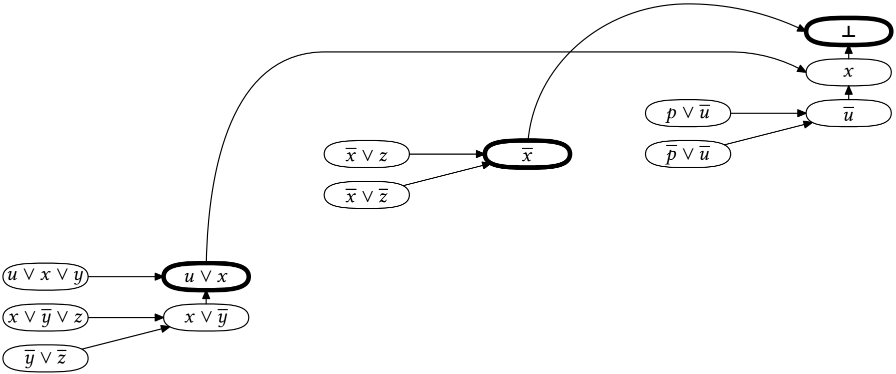

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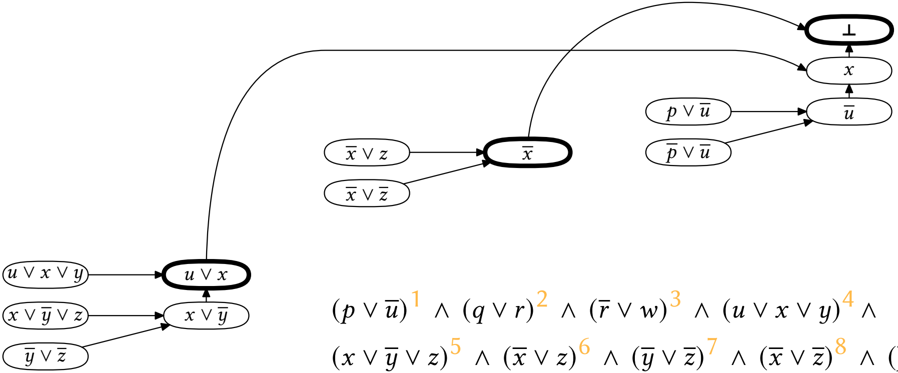

$(p \vee \overline{u})^{1} \wedge (q \vee r)^{2} \wedge (\overline{r} \vee w)^{3} \wedge (u \vee x \vee y)^{4} \wedge$
$(x \vee \overline{y} \vee z)^{5} \wedge (\overline{x} \vee z)^{6} \wedge (\overline{y} \vee \overline{z})^{7} \wedge (\overline{x} \vee \overline{z})^{8} \wedge (\overline{p} \vee \overline{u})^{9}$

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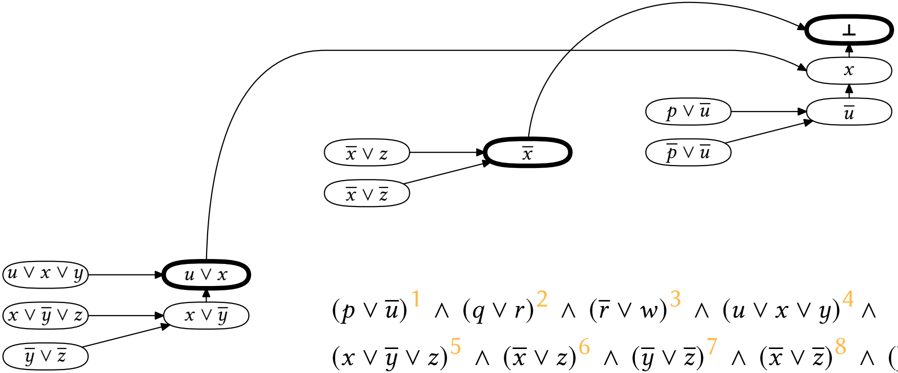


$$(p \vee \overline{u})^1 \wedge (q \vee r)^2 \wedge (\overline{r} \vee w)^3 \wedge (u \vee x \vee y)^4 \wedge$$
$$(x \vee \overline{y} \vee z)^5 \wedge (\overline{x} \vee z)^6 \wedge (\overline{y} \vee \overline{z})^7 \wedge (\overline{x} \vee \overline{z})^8 \wedge (\overline{p} \vee \overline{u})^9$$

```
pol 7 5 + 2 d 4 + 2 d
pol 8 6 + 2 d
pol 9 1 + 2 d 10 + 2 d 11 + 2 d
```

$\rightsquigarrow$ Constraint 10 $\doteq u + x \geq 1$

$\rightsquigarrow$ Constraint 11 $\doteq \overline{x} \geq 1$

$\rightsquigarrow$ Constraint 12 $\doteq 0 \geq 1$ ↯

RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting

Constraint C propagates variable x if setting x to "wrong value" would make C unsatisfiable

E.g., if x_5 is false,

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

would propagate $\overline{x}_4$ (since other coefficients do not add up to 7)

RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting

Constraint C propagates variable x if setting x to "wrong value" would make C unsatisfiable

E.g., if x_5 is false,

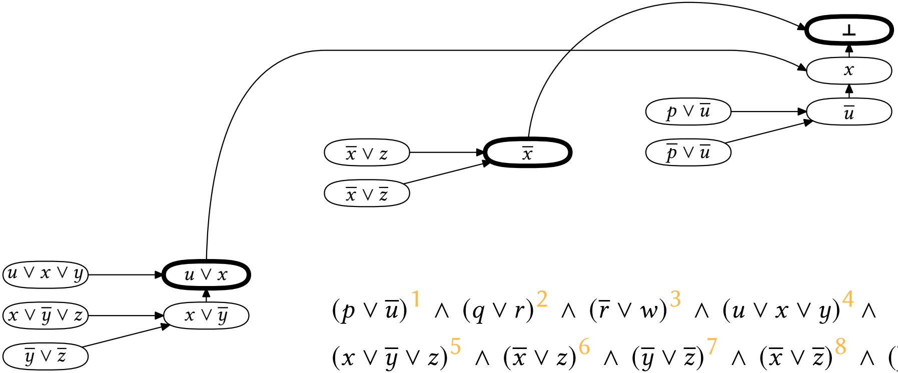
$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

would propagate $\overline{x}_4$ (since other coefficients do not add up to 7)

Risk for confusion:

- Constraint programming people might call this (reverse) integer bounds consistency
  - Does the same thing if we're working with clauses
  - More interesting for general pseudo-Boolean constraints
- SAT people beware: constraints can propagate multiple times and multiple variables

Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



$$(p \lor \overline{u})^{1} \land (q \lor r)^{2} \land (\overline{r} \lor w)^{3} \land (u \lor x \lor y)^{4} \land$$
$$(x \lor \overline{y} \lor z)^{5} \land (\overline{x} \lor z)^{6} \land (\overline{y} \lor \overline{z})^{7} \land (\overline{x} \lor \overline{z})^{8} \land (\overline{p} \lor \overline{u})^{9}$$

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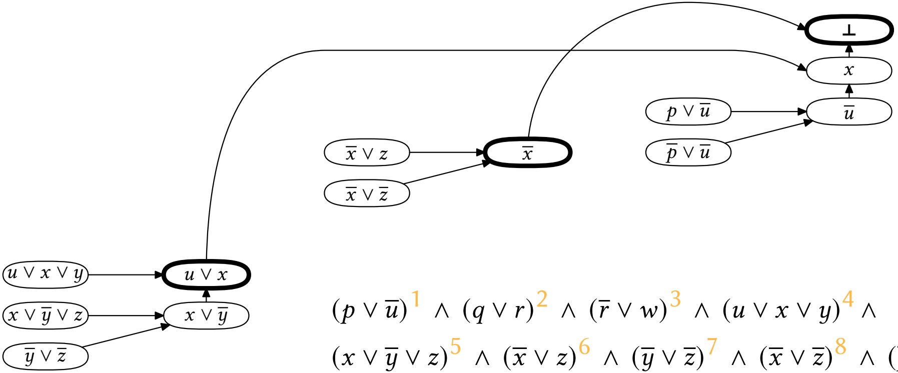

$(p \lor \overline{u})^{1} \land (q \lor r)^{2} \land (\overline{r} \lor w)^{3} \land (u \lor x \lor y)^{4} \land$
$(x \lor \overline{y} \lor z)^{5} \land (\overline{x} \lor z)^{6} \land (\overline{y} \lor \overline{z})^{7} \land (\overline{x} \lor \overline{z})^{8} \land (\overline{p} \lor \overline{u})^{9}$

```
rup 1 u 1 x >= 1 ;
rup 1 ~x >= 1 ;
rup >= 1 ;
```

⇝ Constraint 10 $\doteq u + x \geq 1$
⇝ Constraint 11 $\doteq \overline{x} \geq 1$
⇝ Constraint 12 $\doteq 0 \geq 1$ ↯

Extension Variables, Part 2

Suppose we want new, fresh variable a encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

Again, needs support from the proof system

Proof Logs for “Extended Cutting Planes”

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of pseudo-Boolean constraints in (slight extension of) OPB format [RM16]

- Each constraint follows “obviously” from what is known so far
- Either implicitly, by RUP…
- Or by an explicit cutting planes derivation…
- Or as an extension variable reifying a new constraint*
- Final constraint is $0 \geq 1$

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- Final constraint is $0 \geq 1$

(*) Not actually implemented this way — details in extended version of this tutorial

Deleting Constraints

In practice, important to erase constraints to save memory and time during verification

Fairly straightforward to deal with from the point of view of proof logging

So ignored in this tutorial for simplicity and clarity

Enumeration and Optimisation Problems

Enumeration:

- When a solution is found, can log it
- Introduces a new constraint saying “not this solution”
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Enumeration and Optimisation Problems

Enumeration:

- When a solution is found, can log it
- Introduces a new constraint saying "not this solution"
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For optimisation:

- Define an objective $f = \sum_i w_i \ell_i$, $w_i \in \mathbb{Z}$, to minimise subject to the contraints in the formula
- To maximise, negate objective
- Log a solution α; get an objective-improving constraint $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \alpha(\ell_i)$
- Semantics for proof of optimality: "infeasible to find better solution than best so far"

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0–1 integer linear program (ILP)

- just do proof logging

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- **do not change** input for solver
- **do not change** reasoning in solver
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Goldilocks compromise between expressivity and simplicity:

1. 0–1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments (even for SAT)
3. Efficient reification of constraints

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$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$ $\qquad 7\overline{r} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$ $\qquad 9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$

The VeriPB Format and Tool

https://gitlab.com/MIAOresearch/software/VeriPB

Released under MIT Licence

Various features to help development:

- Extended variable name syntax allowing human-readable names
- Proof tracing
- "Trust me" assertions for incremental proof logging

Documentation:

- Description of VeriPB checker [BMM+23] used in SAT 2023 competition (https://satcompetition.github.io/2023/checkers.html)
- Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23]
- Lots of concrete example files at https://gitlab.com/MIAOresearch/software/VeriPB

Parity (XOR) Reasoning

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
$$\overline{x} \lor y \lor \overline{z}$$
$$\overline{x} \lor \overline{y} \lor z$$

and

$$y \lor z \lor w$$
$$y \lor \overline{z} \lor \overline{w}$$
$$\overline{y} \lor z \lor \overline{w}$$
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want to derive

$$x \lor \overline{w}$$
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$$x + y + z = 1 \pmod 2$$
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$$\overline{y} \vee z \vee \overline{w}$$
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Exponentially hard for CDCL [Urq87]
But used in *CryptoMiniSat* [Cry]

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But used in *CryptoMiniSat* [Cry]

DRAT proof logging like [PR16] too inefficient in practice!

Could add XORs to language, but prefer to keep things super-simple

Pseudo-Boolean Proof Logging for XOR Reasoning

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Introduce extension variables a, b and derive

$$x + y + z + 2a = 3$$
$$y + z + w + 2b = 3$$

(“=” syntactic sugar for “≥” plus “≤”)

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Add to get

$$x + w + 2y + 2z + 2a + 2b = 6$$

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From this can extract

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VeriPB can certify XOR reasoning [GN21]

CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

- *MiniSat+* [ES06]
- *Open-WBO* [MML14]
- *NaPS* [SN15]

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E.g., encode pseudo-Boolean constraint

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

to clauses with extension variables

$$s_{i,k} \Leftrightarrow \sum_{j=1}^{i} x_j \geq k$$

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$\overline{s}_{2,2} \vee x_2$

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$s_{4,2}$

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$$k \cdot \overline{s}_{i,k} + \sum_{j=1}^{i} x_j \geq k$$

$$(i - k + 1) \cdot s_{i,k} + \sum_{j=1}^{i} \overline{x}_j \geq i - k + 1$$

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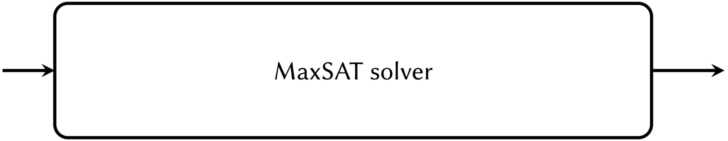
How to know translation is correct?

VeriPB can certify pseudo-Boolean-to-CNF rewriting [GMNO22, VDB22]

Certified Maximum Satisfiability (MaxSAT) Solving

Minimize linear objective subject to satisfying formula in conjunctive normal form (CNF)

$$
\begin{aligned}
\min\ & 2x_1 + x_2 \\
\text{s.t. } & x_1 \vee \overline{z} \\
& z \vee x_2
\end{aligned}
$$



Result:
optimum 1

Many MaxSAT solvers internally make use of SAT solver.

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→ MaxSAT solver → Result: optimum 1

Many MaxSAT solvers internally make use of SAT solver. Idea:

- Find optimal solution (checking that it *is* a solution is easy)
- Add clauses claiming a better solution exists
- Use one extra SAT call to get proof of optimality (with standard SAT proof logging)

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Minimize linear objective subject to satisfying formula in conjunctive normal form (CNF)

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Does not work

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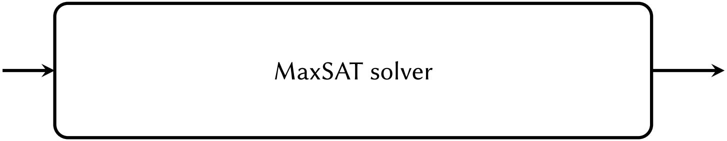

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Many MaxSAT solvers internally make use of SAT solver. Idea:

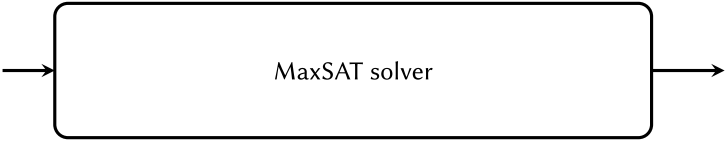
- Find optimal solution (checking that it *is* a solution is easy)
- Add clauses claiming a better solution exists
  Requires proof logging — can be done with VeriPB
- Use one extra SAT call to get proof of optimality (with standard SAT proof logging)
  Causes serious overhead

Does not work

Certified Maximum Satisfiability (MaxSAT) Solving

Minimize linear objective subject to satisfying formula in conjunctive normal form (CNF)

$$\begin{aligned} \min\ & 2x_1 + x_2 \\ \text{s.t. } & x_1 \lor \overline{z} \\ & z \lor x_2 \end{aligned}$$



Result:
optimum 1

Many MaxSAT solvers internally make use of SAT solver. Idea:

- Find optimal solution (checking that it *is* a solution is easy)
- Add clauses claiming a better solution exists
  Requires proof logging — can be done with VeriPB
- Use one extra SAT call to get proof of optimality (with standard SAT proof logging)
  Causes serious overhead

Does not work Only proves answer correct, not reasoning within solver!

MaxSAT Solvers

Three main categories:

- Linear SAT-UNSAT search
  1. Call SAT solver to find some solution
  2. Add clauses encoding "I want a better solution"
  3. Repeat (last found solution is optimal)

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- Implicit Hitting Set
  1. Call SAT solver to find solution under most optimistic assumptions
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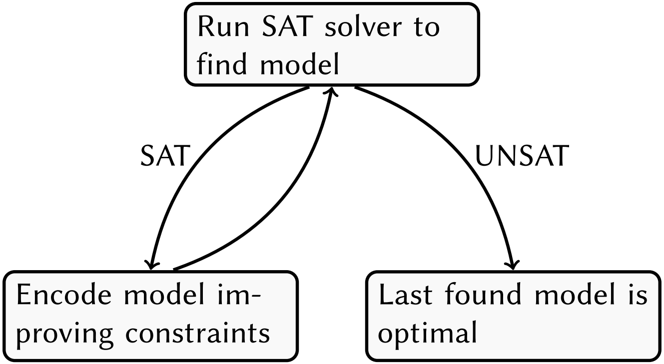
  3. Repeat (first solution is optimal)

  No proof logging available yet

Linear SAT-UNSAT Search



Certified LSU Search (Example)

Objective: $min \sum_i r_i$

VeriPB proof:

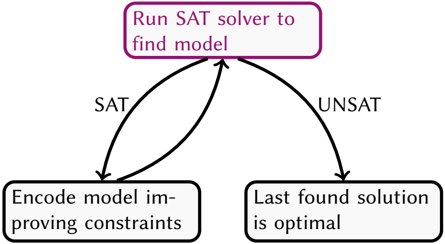
derived	justification

$\overline{x}_1 \lor x_2$ $\quad$ $\overline{x}_1 \lor \overline{x}_2 \lor r_1$

$x_1 \lor \overline{x}_2$ $\quad$ $x_1 \lor x_2 \lor r_2$

$\overline{x}_2 \lor x_3$ $\quad$ $x_2 \lor x_4 \lor r_3$

$\overline{x}_3 \lor x_4$



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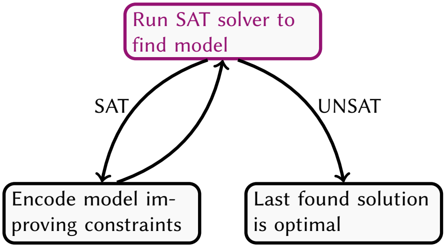
$\overline{x}_3 \vee x_4$

$\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee x_2 \vee r_2$

$x_2 \vee x_4 \vee r_3$

$x_2 \vee r_2$



Certified LSU Search (Example)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation

$\overline{x}_1 \vee x_2$
$x_1 \vee \overline{x}_2$
$\overline{x}_2 \vee x_3$
$\overline{x}_3 \vee x_4$

$\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee x_2 \vee r_2$
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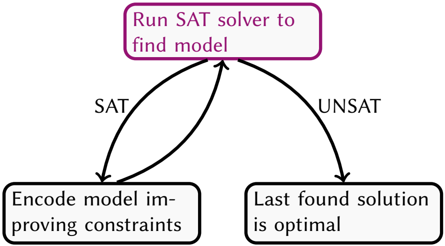

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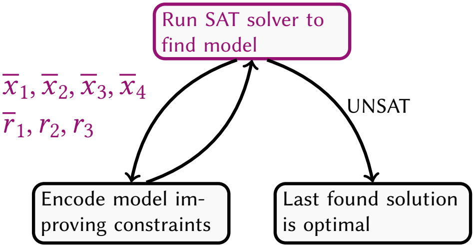

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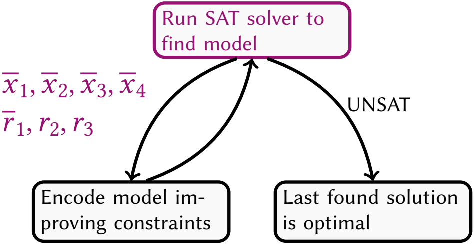

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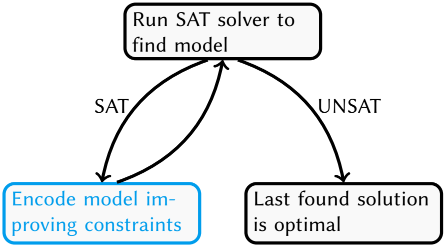

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$\text{PB}(p_2 \Leftrightarrow (\sum_i r_i \geq 2))$	

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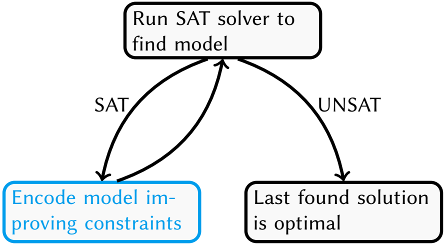

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$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	

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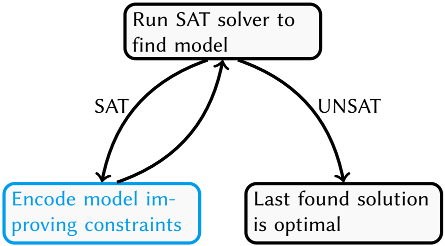

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$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation

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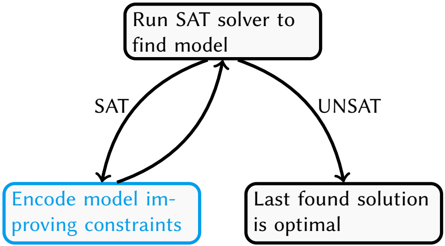

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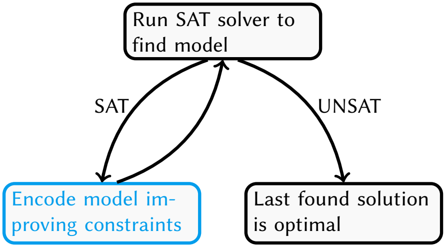

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$\sum_i r_i \leq 1$	Objective Improvement Rule
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$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation

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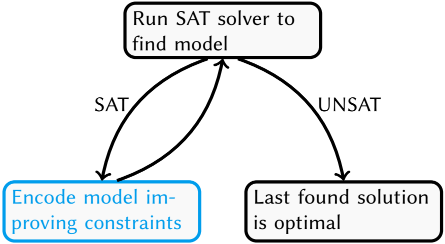

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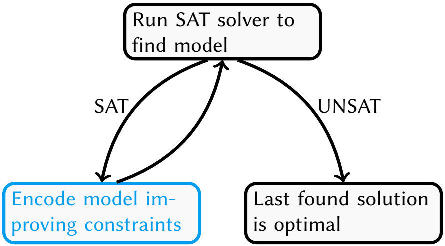
$\overline{x}_3 \vee x_4$ | $x_2 \vee r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$



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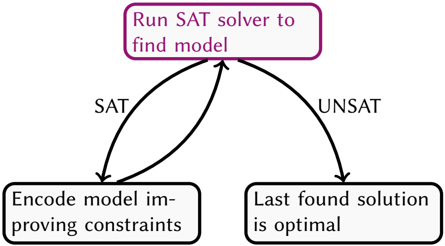

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$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation

$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

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$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$

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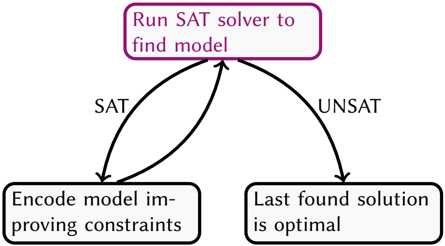
$\overline{p}_2$ $\quad$ x_4



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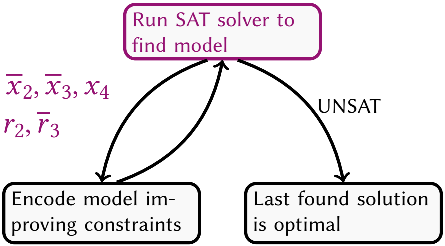
$\overline{p}_2$ $\quad$ x_4

$\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4$

$\overline{r}_1, r_2, \overline{r}_3$



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$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
$\sum_i r_i \leq 0$	Objective Improvement Rule

$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

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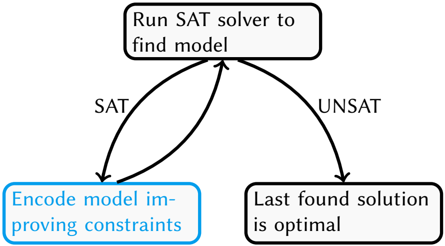
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$ $\quad$ x_4



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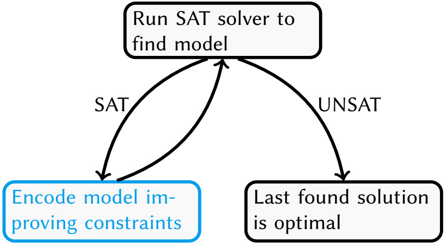
$\sum_i r_i \leq 0$	Objective Improvement Rule
$\overline{p}_1 \geq 1$	Explicit CP derivation

$\overline{x}_1 \vee x_2$	$\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee \overline{x}_2$	$x_1 \vee x_2 \vee r_2$
$\overline{x}_2 \vee x_3$	$x_2 \vee x_4 \vee r_3$
$\overline{x}_3 \vee x_4$	$x_2 \vee r_2$
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	
$\overline{p}_2$	x_4
$\overline{p}_1$	



Certified LSU Search (Example)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
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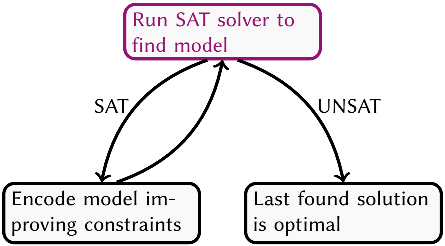
$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$
$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$
$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$
$\overline{p}_2$ $\quad$ x_4
$\overline{p}_1$



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$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
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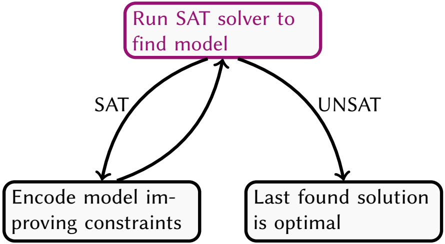
$0 \geq 1$	Reverse Unit Propagation

$\overline{x}_1 \lor x_2$	$\overline{x}_1 \lor \overline{x}_2 \lor r_1$
$x_1 \lor \overline{x}_2$	$x_1 \lor x_2 \lor r_2$
$\overline{x}_2 \lor x_3$	$x_2 \lor x_4 \lor r_3$
$\overline{x}_3 \lor x_4$	$x_2 \lor r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$	x_4
$\overline{p}_1$	$\bot$



Certified LSU Search (Example)

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VeriPB proof:

derived	justification
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$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable
$(4-j) \cdot p_j + \sum_i \overline{r}_i \geq 4-j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
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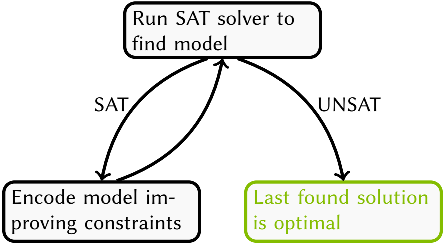
$\overline{x}_1 \vee x_2$	$\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee \overline{x}_2$	$x_1 \vee x_2 \vee r_2$
$\overline{x}_2 \vee x_3$	$x_2 \vee x_4 \vee r_3$
$\overline{x}_3 \vee x_4$	$x_2 \vee r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$	x_4
$\overline{p}_1$	$\perp$



LSU Example in VeriPB Syntax

```
pseudo-Boolean proof version 2.0
f 7
* Clauses derived by solver
rup 1 x1 1 r2 >= 1 ;
* Log incumbent solution
soli ~x1 ~x2 ~x3 ~x4 ~r1 r2 r3
* introduce fresh variables
red 2 ~p2 1 r1 1 r2 1 r3  >= 2 ; p2 -> 0 ;
red 2 p2 1 ~r1 1 ~r2 1 ~r3  >= 2; p2 -> 1 ;
red 1 ~p1 1 r1 1 r2 1 r3  >= 1; p1 -> 0 ;
red 3 p1 1 ~r1 1 ~r2 1 ~r3 >= 3; p1 -> 1 ;
* Derive CNF encoding of totalizer
. . .  - coming soon
* Derive counter falsity
pol 9 10 + s
* Clauses derived by solver
rup 1 x4 >= 1 ;
* Log incumbent solution
soli ~x1 ~x2 ~x3 x4 ~r1 r2 ~r3
* Derive counter falsity
pol -1 12 +
* Inconsistency derived by solver
rup >= 1 ;
* Conclusion
output NONE
conclusion BOUNDS 1 1
end pseudo-Boolean proof
```

Certified Encoding of the Model-Improving Constraint

How to encode $p_j \Leftrightarrow \sum_i r_i \geq j$ in CNF?

Certified Encoding of the Model-Improving Constraint

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Different MaxSAT solvers use different PB-to-CNF encodings, e.g.,

- Totalizer Encoding [BB03]
- Binary Adder [War98]
- Modulo-Based Totalizer [OLH+13]
- Sorting Networks [ES06, ANOR09]
- (Dynamic) Polynomial Watchdog [PRB18]

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- Sorting Networks [ES06, ANOR09]
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Totalizer encoding demonstrated here; ideas generalize to other encodings [Van23]

Totalizer Encoding of Cardinality Constraints

How to encode $p_j^I \Leftrightarrow \sum_{i \in I} r_i \geq j$?

- Totalizer encoding [BB03]

Totalizer Encoding of Cardinality Constraints

How to encode $p^I_j \Leftrightarrow \sum_{i \in I} r_i \geq j$?

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- Create binary tree (leaves are the r_i); and introduce counter variables in all nodes

Totalizer Encoding of Cardinality Constraints

How to encode $p_j^I \Leftrightarrow \sum_{i \in I} r_i \geq j$?

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- Create binary tree (leaves are the r_i); and introduce counter variables in all nodes
- Example: $I = \{1, \cdots, 8\}$, $I_1 = \{1, \cdots, 4\}$ and $I_2 = \{5, \cdots, 8\}$

$$p_1^I, p_2^I, p_3^I, p_4^I, p_5^I, p_6^I, p_7^I, p_8^I$$

$$p_1^{I_1}, p_2^{I_1}, p_3^{I_1}, p_4^{I_1} \qquad\qquad p_1^{I_2}, p_2^{I_2}, p_3^{I_2}, p_4^{I_2}$$

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How to encode $p^I_j \Leftrightarrow \sum_{i \in I} r_i \geq j$?

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- Example: $I = \{1, \cdots, 8\}$, $I_1 = \{1, \cdots, 4\}$ and $I_2 = \{5, \cdots, 8\}$

$$p^I_1, p^I_2, p^I_3, p^I_4, p^I_5, p^I_6, p^I_7, p^I_8$$

$$p^{I_1}_1, p^{I_1}_2, p^{I_1}_3, p^{I_1}_4 \qquad p^{I_2}_1, p^{I_2}_2, p^{I_2}_3, p^{I_2}_4$$

Clauses encoding $p^I_6 \Leftarrow \sum_{i \in I} r_i \geq 6$:

$$\left(p^{I_1}_2 \wedge p^{I_2}_4\right) \Rightarrow p^I_6 \qquad \left(p^{I_1}_3 \wedge p^{I_2}_3\right) \Rightarrow p^I_6 \qquad \left(p^{I_1}_4 \wedge p^{I_2}_2\right) \Rightarrow p^I_6$$

Totalizer Encoding of Cardinality Constraints

How to encode $p^I_j \Leftrightarrow \sum_{i \in I} r_i \geq j$?

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- Example: $I = \{1, \cdots, 8\}$, $I_1 = \{1, \cdots, 4\}$ and $I_2 = \{5, \cdots, 8\}$

$$p^I_1, p^I_2, p^I_3, p^I_4, p^I_5, p^I_6, p^I_7, p^I_8$$

$$p^{I_1}_1, p^{I_1}_2, p^{I_1}_3, p^{I_1}_4 \qquad p^{I_2}_1, p^{I_2}_2, p^{I_2}_3, p^{I_2}_4$$

Clauses encoding $p^I_6 \Leftarrow \sum_{i \in I} r_i \geq 6$:

$$\overline{p}^{I_1}_2 \vee \overline{p}^{I_2}_4 \vee p^I_6 \qquad \overline{p}^{I_1}_3 \vee \overline{p}^{I_2}_3 \vee p^I_6 \qquad \overline{p}^{I_1}_4 \vee \overline{p}^{I_2}_2 \vee p^I_6$$

Totalizer Encoding of Cardinality Constraints

How to encode $p^I_j \Leftrightarrow \sum_{i \in I} r_i \geq j$?

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- Create binary tree (leaves are the r_i); and introduce counter variables in all nodes
- Example: $I = \{1, \cdots, 8\}$, $I_1 = \{1, \cdots, 4\}$ and $I_2 = \{5, \cdots, 8\}$

$$p^I_1, p^I_2, p^I_3, p^I_4, p^I_5, p^I_6, p^I_7, p^I_8$$

$$p^{I_1}_1, p^{I_1}_2, p^{I_1}_3, p^{I_1}_4 \qquad\qquad p^{I_2}_1, p^{I_2}_2, p^{I_2}_3, p^{I_2}_4$$

Clauses encoding $p^I_6 \Leftarrow \sum_{i \in I} r_i \geq 6$:

$$\overline{p}^{I_1}_2 \vee \overline{p}^{I_2}_4 \vee p^I_6 \qquad \overline{p}^{I_1}_3 \vee \overline{p}^{I_2}_3 \vee p^I_6 \qquad \overline{p}^{I_1}_4 \vee \overline{p}^{I_2}_2 \vee p^I_6$$

Clauses encoding $p^I_6 \Rightarrow \sum_{i \in I} r_i \geq 6$:

$$\overline{p}^{I_1}_2 \Rightarrow \overline{p}^I_6 \qquad \left(\overline{p}^{I_1}_3 \wedge \overline{p}^{I_2}_4\right) \Rightarrow \overline{p}^I_6 \qquad \left(\overline{p}^{I_1}_4 \wedge \overline{p}^{I_2}_3\right) \Rightarrow \overline{p}^I_6 \qquad \overline{p}^{I_2}_2 \Rightarrow \overline{p}^I_6$$

Totalizer Encoding of Cardinality Constraints

How to encode $p^I_j \Leftrightarrow \sum_{i \in I} r_i \geq j$?

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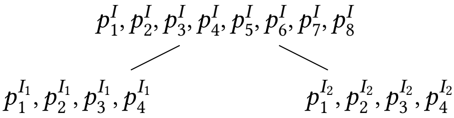
- Create binary tree (leaves are the r_i); and introduce counter variables in all nodes
- Example: $I = \{1, \cdots, 8\}$, $I_1 = \{1, \cdots, 4\}$ and $I_2 = \{5, \cdots, 8\}$



Clauses encoding $p^I_6 \Leftarrow \sum_{i \in I} r_i \geq 6$:

$$\overline{p}^{I_1}_2 \vee \overline{p}^{I_2}_4 \vee p^I_6 \qquad \overline{p}^{I_1}_3 \vee \overline{p}^{I_2}_3 \vee p^I_6 \qquad \overline{p}^{I_1}_4 \vee \overline{p}^{I_2}_2 \vee p^I_6$$

Clauses encoding $p^I_6 \Rightarrow \sum_{i \in I} r_i \geq 6$:

$$p^{I_1}_2 \vee \overline{p}^I_6 \qquad p^{I_1}_3 \vee p^{I_2}_4 \vee \overline{p}^I_6 \qquad p^{I_1}_4 \vee p^{I_2}_3 \vee \overline{p}^I_6 \qquad p^{I_2}_2 \vee \overline{p}^I_6$$

Certifying the Totalizer encoding using cutting planes

- To be derived: $\overline{p}_4^{I_1} \vee \overline{p}_2^{I_2} \vee p_6^{I}$

Certifying the Totalizer encoding using cutting planes

- To be derived: $\overline{p}_4^{I_1} \lor \overline{p}_2^{I_2} \lor p_6^{I}$
- Counting variables introduced using

$$4 \cdot \overline{p}_4^{I_1} + \sum_{i \in I_1} r_i \geq 4$$

$$2 \cdot \overline{p}_2^{I_2} + \sum_{i \in I_2} r_i \geq 2$$

$$3 \cdot p_6^{I} + \sum_{i \in I} \overline{r}_i \geq 3$$

Certifying the Totalizer encoding using cutting planes

- To be derived: $\overline{p}_4^{I_1} \vee \overline{p}_2^{I_2} \vee p_6^{I}$
- Counting variables introduced using

$$4 \cdot \overline{p}_4^{I_1} + \sum_{i \in I_1} r_i \geq 4$$

$$2 \cdot \overline{p}_2^{I_2} + \sum_{i \in I_2} r_i \geq 2$$

$$3 \cdot p_6^{I} + \sum_{i \in I} \overline{r}_i \geq 3$$

- Adding these three constraints yields

$$4 \cdot \overline{p}_4^{I_1} + 2 \cdot \overline{p}_2^{I_2} + 3 \cdot p_6^{I} + 8 \geq 9$$

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- To be derived: $\overline{p}_4^{I_1} \vee \overline{p}_2^{I_2} \vee p_6^{I}$
- Counting variables introduced using

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- Adding these three constraints yields

$$4 \cdot \overline{p}_4^{I_1} + 2 \cdot \overline{p}_2^{I_2} + 3 \cdot p_6^{I} + \not{8} \geq \not{9}\ 1$$

Certifying the Totalizer encoding using cutting planes

- To be derived: $\overline{p}_4^{I_1} \vee \overline{p}_2^{I_2} \vee p_6^{I}$
- Counting variables introduced using

$$4 \cdot \overline{p}_4^{I_1} + \sum_{i \in I_1} r_i \geq 4$$

$$2 \cdot \overline{p}_2^{I_2} + \sum_{i \in I_2} r_i \geq 2$$

$$3 \cdot p_6^{I} + \sum_{i \in I} \overline{r}_i \geq 3$$

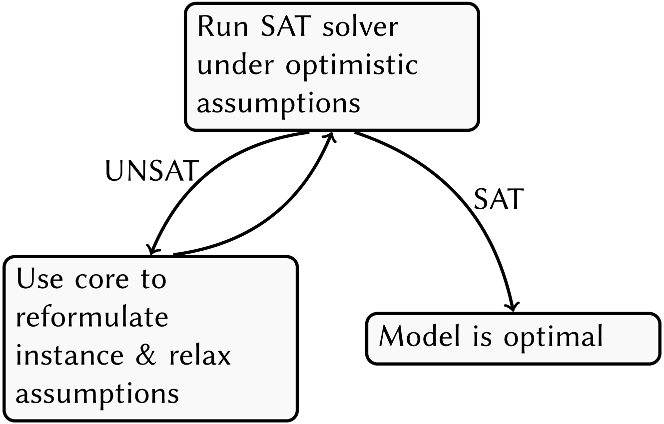
- Adding these three constraints and saturating yields

$$\cancel{4} \cdot \overline{p}_4^{I_1} + \cancel{2} \cdot \overline{p}_2^{I_2} + \cancel{3} \cdot p_6^{I} + \cancel{8} \geq \cancel{9}\ 1$$

Complete LSU Example in VeriPB Syntax

```
pseudo-Boolean proof version 2.0
f 7
* Clauses derived by solver
rup 1 x1 1 r2 >= 1 ;
* Log incumbent solution
soli ~x1 ~x2 ~x3 ~x4 ~r1 r2 r3
* introduce fresh variables
red 2 ~p2 1 r1 1 r2 1 r3  >= 2 ; p2 -> 0 ;
red 2 p2 1 ~r1 1 ~r2 1 ~r3  >= 2; p2 -> 1 ;
red 1 ~p1 1 r1 1 r2 1 r3  >= 1; p1 -> 0 ;
red 3 p1 1 ~r1 1 ~r2 1 ~r3 >= 3; p1 -> 1 ;
* Auxiliary variables for CNF encoding
red 2 ~p_1-2_2 1 r1 1 r2  >= 2 ; p_1-2_2 -> 0 ;
red 1 p_1-2_2 1 ~r1 1 ~r2 >= 1; p_1-2_2 -> 1 ;
red 1 ~p_1-2_1 1 r1 1 r2  >= 1; p_1-2_1 -> 0 ;
red 2 p_1-2_1 1 ~r1 1 ~r2 >= 2; p_1-2_1 -> 1 ;
* Cutting planes derivation of totalizer clauses
pol 10 15 + s
pol 10 17 + ~r3 + s
pol 11 14 + r3 + s
pol 11 16 + s
pol 12 17 + s
pol 13 16 + r3 + s
pol 13 r1 + r2 + s
* Derive counter falsity
pol 9 10 + s
* Clauses derived by solver
rup 1 x4 >= 1 ;
* Log incumbent solution
soli ~x1 ~x2 ~x3 x4 ~r1 r2 ~r3
* Derive counter falsity
pol -1 12 +
* Inconsistency derived by solver
rup >= 1 ;
* Conclusion
output NONE
conclusion BOUNDS 1 1
end pseudo-Boolean proof
```

Core-Guided Search



Certified Core-Guided Search (Example)

$\overline{x}_1 \lor x_2$ $\quad$ $\overline{x}_1 \lor \overline{x}_2 \lor r_1$

$x_1 \lor \overline{x}_2$ $\quad$ $x_1 \lor x_2 \lor r_2$

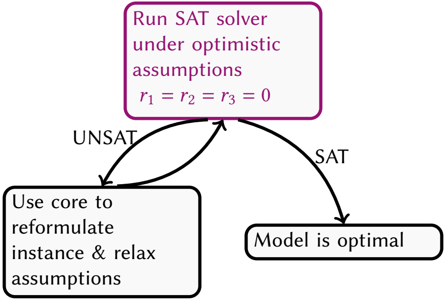
$\overline{x}_2 \lor x_3$ $\quad$ $x_2 \lor x_4 \lor r_3$

$\overline{x}_3 \lor x_4$

Objective (*min*): $r_1 + r_2 + r_3$

VeriPB proof:

derived	justification



Certified Core-Guided Search (Example)

Objective (*min*): $r_1 + r_2 + r_3$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
Core returned by solver:	
$r_1 + r_2 \geq 1$	Reverse Unit Propagation

$\overline{x}_1 \lor x_2$

$x_1 \lor \overline{x}_2$

$\overline{x}_2 \lor x_3$

$\overline{x}_3 \lor x_4$

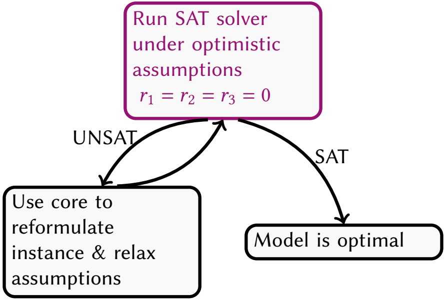
$r_1 \lor r_2$

$\overline{x}_1 \lor \overline{x}_2 \lor r_1$

$x_1 \lor x_2 \lor r_2$

$x_2 \lor x_4 \lor r_3$

$x_2 \lor r_2$



Certified Core-Guided Search (Example)

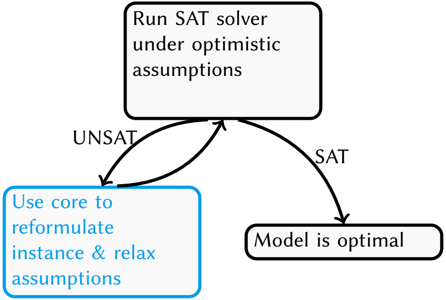
Objective (*min*): $r_1 + r_2 + r_3$

$\overline{x}_1 \vee x_2$
$x_1 \vee \overline{x}_2$
$\overline{x}_2 \vee x_3$
$\overline{x}_3 \vee x_4$
$r_1 \vee r_2$

$\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee x_2 \vee r_2$
$x_2 \vee x_4 \vee r_3$
$x_2 \vee r_2$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
Core returned by solver:	
$r_1 + r_2 \geq 1$	Reverse Unit Propagation
$p_2 \Leftrightarrow (r_1 + r_2 \geq 2)$	Fresh variable



Certified Core-Guided Search (Example)

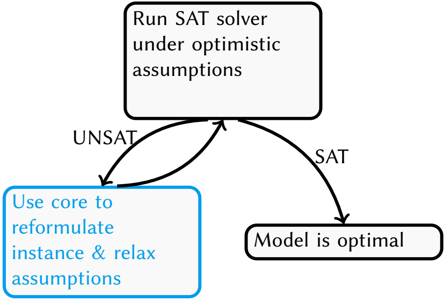
$\overline{x}_1 \vee x_2$
$x_1 \vee \overline{x}_2$
$\overline{x}_2 \vee x_3$
$\overline{x}_3 \vee x_4$
$r_1 \vee r_2$

$\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee x_2 \vee r_2$
$x_2 \vee x_4 \vee r_3$
$x_2 \vee r_2$

Objective (*min*): $r_1 + r_2 + r_3$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
Core returned by solver:	
$r_1 + r_2 \geq 1$	Reverse Unit Propagation
$2 \cdot \overline{p}_2 + r_1 + r_2 \geq 2$	Fresh variable
$p_2 + \overline{r}_1 + \overline{r}_2 \geq 1$	



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Objective (*min*): $r_1 + r_2 + r_3$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
Core returned by solver:	
$r_1 + r_2 \geq 1$	Reverse Unit Propagation
$2 \cdot \overline{p}_2 + r_1 + r_2 \geq 2$	Fresh variable
$p_2 + \overline{r}_1 + \overline{r}_2 \geq 1$	
$\mathrm{CNF}(p_2 \Leftrightarrow (r_1 + r_2 \geq 2))$	Explicit CP derivation

$\overline{x}_1 \vee x_2$

$x_1 \vee \overline{x}_2$

$\overline{x}_2 \vee x_3$

$\overline{x}_3 \vee x_4$

$r_1 \vee r_2$

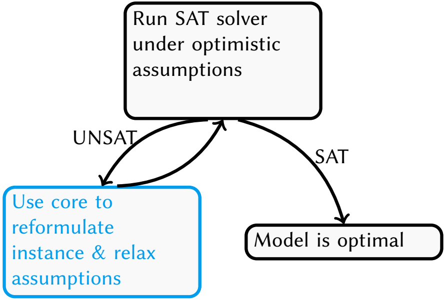
$\mathrm{CNF}(p_2 \Leftrightarrow (r_1 + r_2 \geq 2))$

$\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee x_2 \vee r_2$

$x_2 \vee x_4 \vee r_3$

$x_2 \vee r_2$



Certified Core-Guided Search (Example)

Objective (*min*): $r_1 + r_2 + r_3 = 1 + p_2 + r_3$

VeriPB proof:

derived	justification
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Core returned by solver:	
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$p_2 + \overline{r}_1 + \overline{r}_2 \geq 1$	
$\mathrm{CNF}(p_2 \Leftrightarrow (r_1 + r_2 \geq 2))$	Explicit CP derivation
$r_1 + r_2 = 1 + p_2$	Explicit CP derivation

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$x_1 \vee \overline{x}_2$

$\overline{x}_2 \vee x_3$

$\overline{x}_3 \vee x_4$

$r_1 \vee r_2$

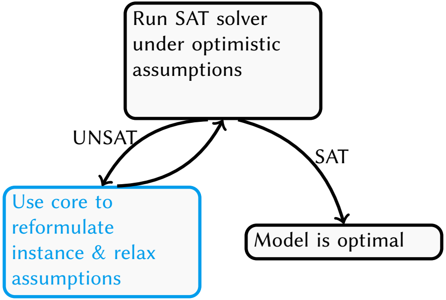
$\mathrm{CNF}(p_2 \Leftrightarrow (r_1 + r_2 \geq 2))$

$\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee x_2 \vee r_2$

$x_2 \vee x_4 \vee r_3$

$x_2 \vee r_2$



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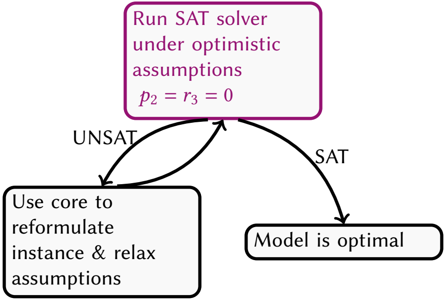

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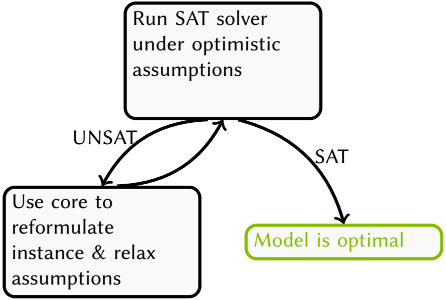

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$\overline{r}_1 + \overline{r}_2 + \overline{r}_3 \geq 3$	Objective Improvement

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$r_1 \lor r_2$

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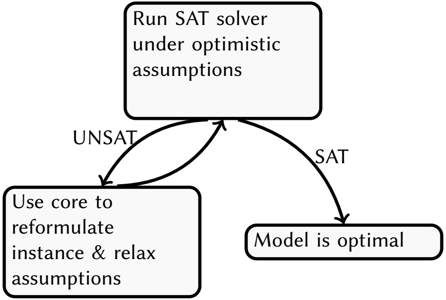

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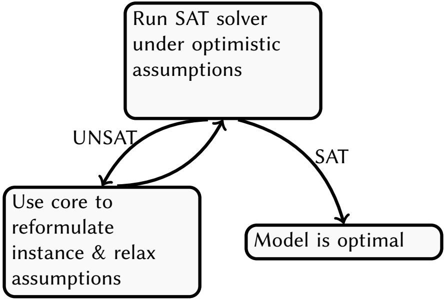

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CNF encoding (totalizer): see part on LSU

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Adding up definition of p_2 and core constraint yields

$$2 \cdot \overline{p}_2 + 2 \cdot r_1 + 2 \cdot r_2 \geq 3 \;.$$

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Explicit CP derivations:

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Adding up definition of p_2 and core constraint and dividing by 2 yields

$$\cancel{2} \cdot \overline{p}_2 + \cancel{2} \cdot r_1 + \cancel{2} \cdot r_2 \geq \cancel{3}2.$$

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Previously derived cores guarantee that objective is at least 1:
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Previously derived cores guarantee that objective is at least 1:
$r_1 + r_2 \ (+ r_3) \geq 1$
Adding this to objective improvement constraint gives contradiction

Complete CG Example in VeriPB Syntax

```
pseudo-Boolean proof version 2.0
f 7
* Clauses derived by solver (inc core)
rup 1 x1 1 r2 >= 1 ;
rup 1 r1 1 r2 >= 1 ;
* Introduce fresh variable
red 2 ~p2 1 r1 1 r2  >= 2 ; p2 -> 0 ;
red 1 p2 1 ~r1 1 ~r2 >= 1; p2 -> 1 ;
* Encode this in CNF
pol 10 ~r1 +
pol 10 ~r2 +
* Rewriting the objective
pol 9 10 + 2 d
* Check that we have indeed
*   derived that r1 + r2 = 1 + p2
e 14 : 1 r1 1 r2 -1 p2 >= 1 ;
e 11 : -1 r1 -1 r2 1 p2 >= -1 ;
```

```
* Solution found
soli x1 x2 x3 x4 r1 ~r2 ~r3
* Prove optimality of solution:
pol -1 9 +
ia -1 : >= 1 ;
* Conclusion
output NONE
conclusion BOUNDS 1 1
end pseudo-Boolean proof
```

Advanced Techniques for Core-Guided MaxSAT

- Important to deal with all state-of-the-art solver techniques

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  - Lazy counter variables [MJML14]

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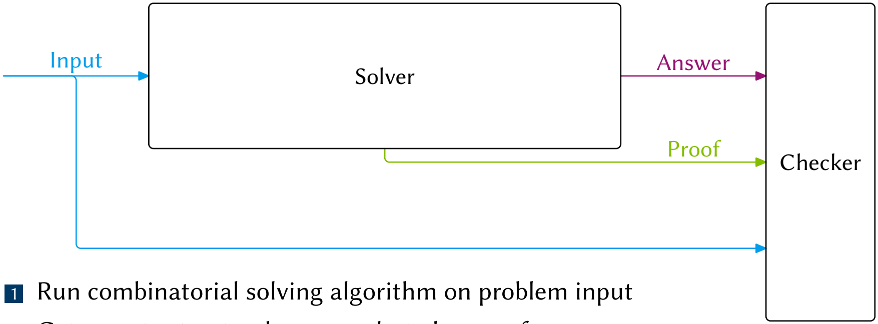
- Additional techniques that are skipped in this example
  - Intrinsic at-most-one constraints [IMM19]
  - Hardening [ABGL12]
  - Lazy counter variables [MJML14]
- VeriPB Proof logging also convenient for these techniques [BBN+23]

Recap (1/2)



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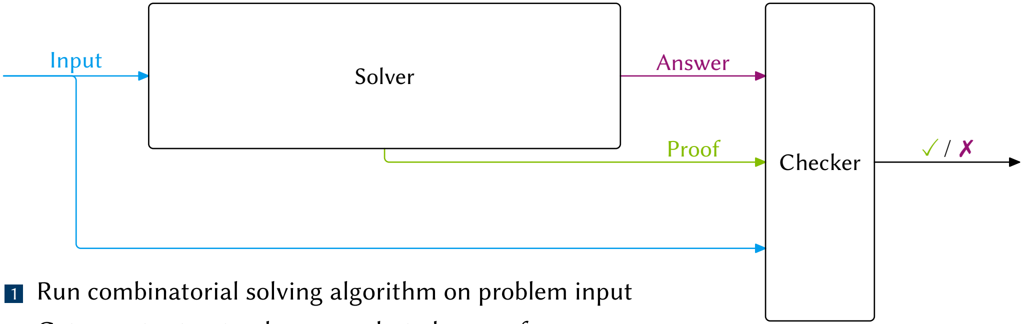

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Recap (2/2)

Proof logging implementation

- Don't change solver
- Just add proof logging statements (plus some book-keeping)

Performance goals

Want linear(ish) scaling in terms of solver running time for

- proof size
- proof checking time

Progress So Far

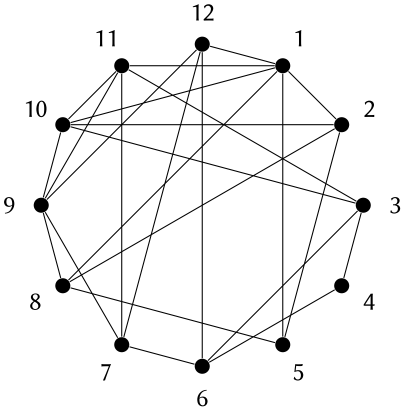
We've seen proof logging, and how it works for SAT

We've learned about

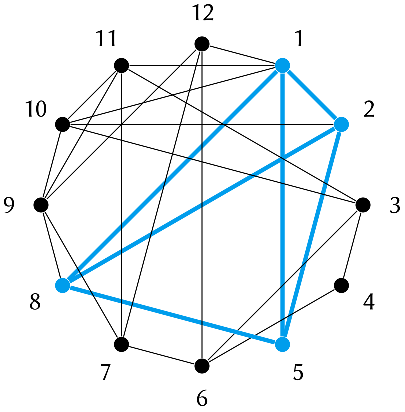
- pseudo-Boolean constraints (0–1 linear inequalities)
- cutting planes reasoning
- VeriPB

Coming next, some worked examples from dedicated graph solvers

The Maximum Clique Problem



The Maximum Clique Problem



Maximum Clique Solvers

There are a lot of dedicated solvers for clique problems

But there are issues:

- “State-of-the-art” solvers have been buggy.
- Often undetected: error rate of around 0.1 [MPP19]

Often used inside other solvers

- An off-by-one result can cause much larger errors

A Brief and Incomplete Guide to Clique Solving (1/4)

Recursive maximum clique algorithm:

- Pick a vertex v
- Either v is in the clique…
  - Throw away every vertex not adjacent to v
  - If vertices remain, recurse
- …or v is not in the clique
  - Throw v away and pick another vertex

A Brief and Incomplete Guide to Clique Solving (2/4)

Key data structures:

- Growing clique C
- Set of potential vertices P
  - All the vertices we haven't thrown away yet
  - Every $v \in P$ is adjacent to every $w \in C$

A Brief and Incomplete Guide to Clique Solving (2/4)

Key data structures:

- Growing clique C
- Set of potential vertices P
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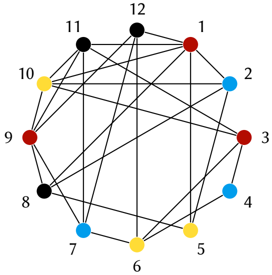
  - Every $v \in P$ is adjacent to every $w \in C$

Branch and bound:

- Remember the biggest clique $C^\star$ found so far
- If $|C| + |P| \leq \left|C^\star\right|$, no need to keep going

A Brief and Incomplete Guide to Clique Solving (3/4)



Given a k-colouring of a subgraph, that subgraph cannot have a clique of more than k vertices

We can use $|C| + \#colours(P)$ as a bound, for any colouring

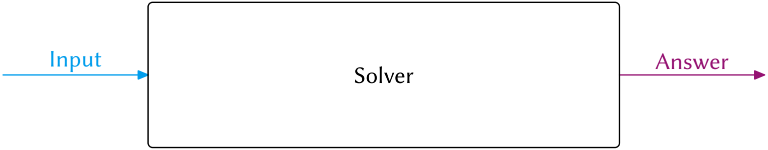
A Brief and Incomplete Guide to Clique Solving (4/4)

- This brings us to 1997
- Many improvements since then
  - better bound functions
  - clever vertex selection heuristics
  - efficient data structures
  - local search
  - ...
- But key ideas for proof logging can be explained without worrying about such things

Making a Proof Logging Clique Solver

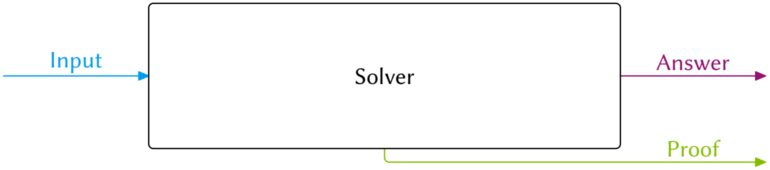
1. Output a pseudo-Boolean encoding of the problem
   - Clique problems have several standard file formats
2. Make the solver log its search tree
   - Output a small header
   - Output something on every backtrack
   - Output something every time a solution is found
   - Output a small footer
3. Figure out how to log the bound function

A Slightly Different Proof Logging Workflow



1. Run combinatorial solving algorithm on problem input

A Slightly Different Proof Logging Workflow



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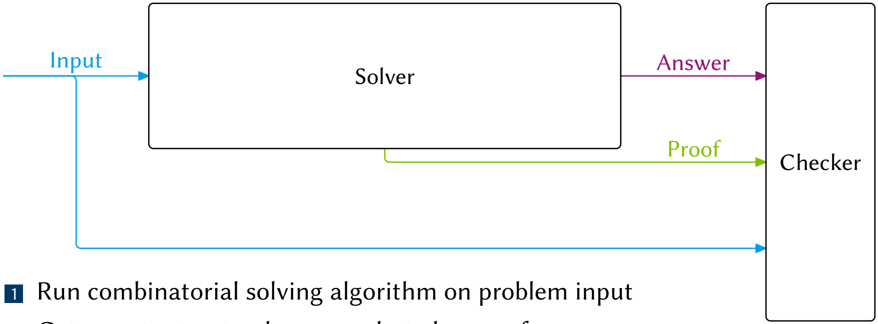

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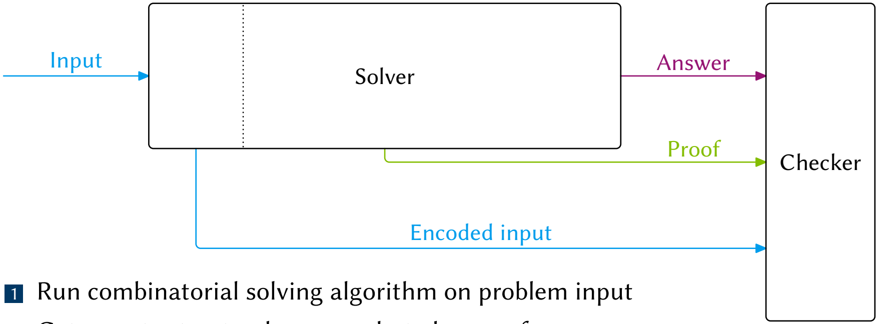

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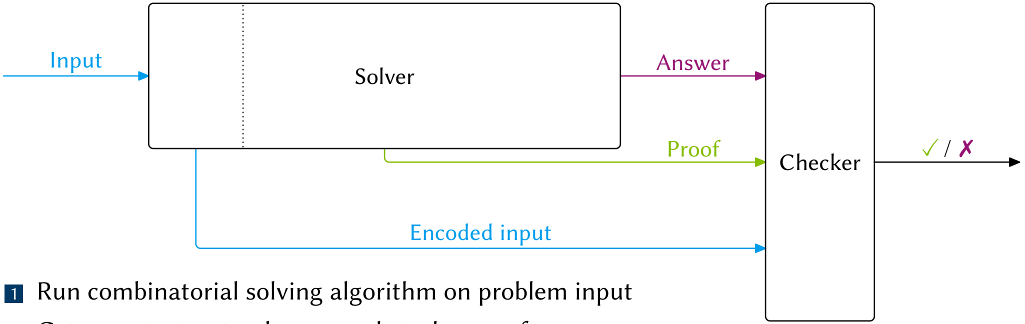

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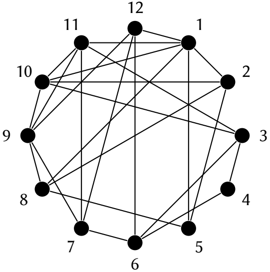
4. Verify that proof checker says answer is correct

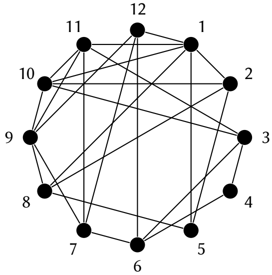
A Pseudo-Boolean Encoding for Clique (in OPB Format)




```
* #variable= 12 #constraint= 41
min: -1 x1 -1 x2 -1 x3 -1 x4 . . . and so on. . . -1 x11 -1 x12 ;
1 ~x3 1 ~x1 >= 1 ;
1 ~x3 1 ~x2 >= 1 ;
1 ~x4 1 ~x1 >= 1 ;
* . . . and a further 38 similar lines for the remaining non-edges
```

First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
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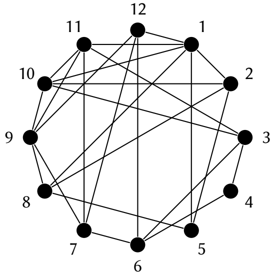

Start with a header
Load the 41 problem axioms

First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
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rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
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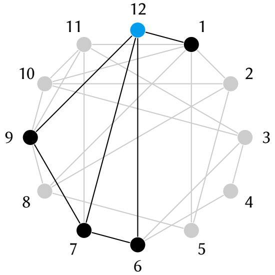

Branch accepting 12
Throw away non-adjacent vertices

First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
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end pseudo-Boolean proof
```
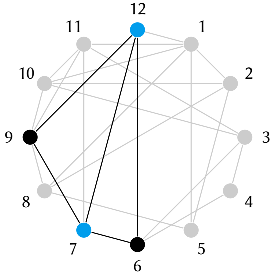
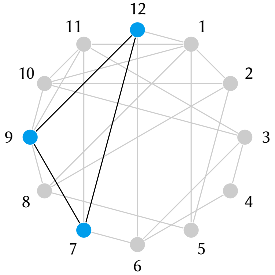



Branch also accepting 7
Throw away non-adjacent vertices

First Attempt at a Proof

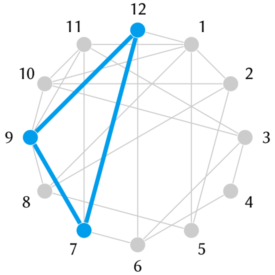
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch also accepting 9
Throw away non-adjacent vertices

First Attempt at a Proof

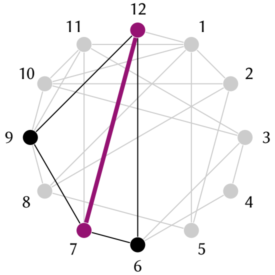
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



We branched on 12, 7, 9
Found a new incumbent
Now looking for a ≥ 4 vertex clique

First Attempt at a Proof

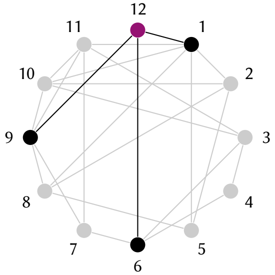
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 12, 7
9 explored already, only 6 feasible
No ≥ 4 vertex clique possible
Effectively this deletes the 7–12 edge

First Attempt at a Proof

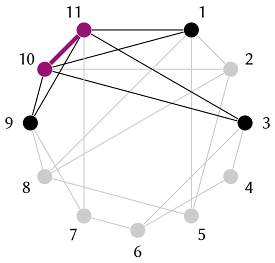
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 12
Only 1, 6 and 9 feasible (1-colourable)
No ≥ 4 vertex clique possible
Effectively this deletes vertex 12

First Attempt at a Proof

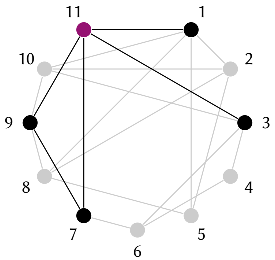
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch on 11 then 10
Only 1, 3 and 9 feasible (1-colourable)
No ≥ 4 vertex clique possible
Backtrack, deleting the edge

First Attempt at a Proof

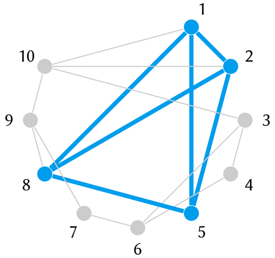
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 11
2-colourable, so no ≥ 4 clique
Delete the vertex

First Attempt at a Proof

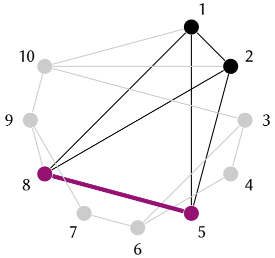
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch on 8, 5, 1, 2
Find a new incumbent
Now looking for a ≥ 5 vertex clique

First Attempt at a Proof

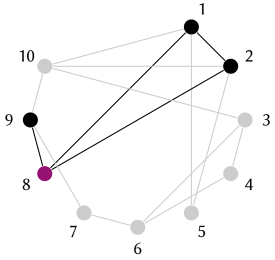
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 8, 5
Only 4 vertices; can't have a ≥ 5 clique
Delete the edge

First Attempt at a Proof

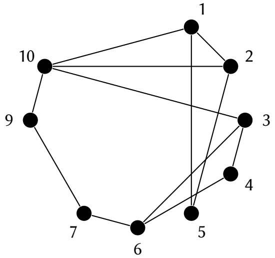
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 8
Still not enough vertices
Delete the vertex

First Attempt at a Proof

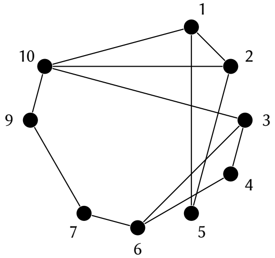
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Remaining graph is 3-colourable
Backtrack from root node

First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Finish with what we've concluded
We specify a lower and an upper bound
Remember we're minimising $\sum_v -1 \times v$, so a 4-clique has an objective value of -4

Verifying This Proof (Or Not...)

```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

Verifying This Proof (Or Not. . .)

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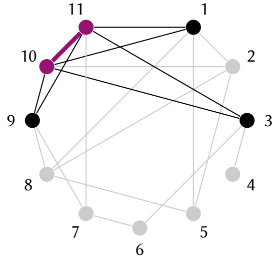

Verifying This Proof (Or Not…)

```
$ veripb --trace clique.opb clique-attempt-one.veripb
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: soli x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: rup 1 ~x12 >= 1 ;
  ConstraintId 044: 1 ~x12 >= 1
line 006: rup 1 ~x11 1 ~x10 >= 1 ;
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

Dealing With Colourings

The colour bound doesn't follow by RUP…

But we can lazily recover at-most-one constraints for each colour class!

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$$
\begin{aligned}
&(\overline{x}_1 + \overline{x}_6 \geq 1) && \\
+&(\overline{x}_1 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
+&(\overline{x}_6 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + 2\overline{x}_6 + 2\overline{x}_9 \geq 3 \\
&/\,2 && = \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& && \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}
$$

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(\overline{x}_1 + \overline{x}_6 \geq 1) & \\
+ (\overline{x}_1 + \overline{x}_9 \geq 1) & \quad = 2\overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
+ (\overline{x}_6 + \overline{x}_9 \geq 1) & \quad = 2\overline{x}_1 + 2\overline{x}_6 + 2\overline{x}_9 \geq 3 \\
/\, 2 & \quad = \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& \quad \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}
$$

This generalises to colour classes of any size v

- Each non-edge is used exactly once, $v(v-1)$ additions
- $v-3$ multiplications and $v-2$ divisions

Solvers don't need to "understand" cutting planes to write this derivation to proof log

What This Looks Like in the Proof Log

pseudo-Boolean proof version 2.0
f 41
soli x12 x7 x9
rup 1 ~x12 1 ~x7 >= 1 ;
* bound, colour classes [x1 x6 x9]
pol $7_{1 \not\sim 6}$ $19_{1 \not\sim 9}$ + $24_{6 \not\sim 9}$ + 2 d
pol 42_{obj} -1 +
rup 1 ~x12 >= 1 ;
* bound, colour classes [x1 x3 x9]
pol $1_{1 \not\sim 3}$ $19_{1 \not\sim 9}$ + $21_{3 \not\sim 9}$ + 2 d
pol 42_{obj} -1 +
rup 1 ~x11 1 ~x10 >= 1 ;
* bound, colour classes [x1 x3 x7]
* [x9]
pol $1_{1 \not\sim 3}$ $10_{1 \not\sim 7}$ + $12_{3 \not\sim 7}$ + 2 d
pol 42_{obj} -1 +
rup 1 ~x11 >= 1 ;
soli x8 x5 x2 x1
rup 1 ~x8 1 ~x5 >= 1 ;
* bound, colour classes [x1 x9] [x2]
pol 53_{obj} $19_{1 \not\sim 9}$ +
rup 1 ~x8 >= 1 ;
* bound, colour classes [x1 x3 x7]
* [x2 x4 x9] [x5 x6 x10]
pol $1_{1 \not\sim 3}$ $10_{1 \not\sim 7}$ + $12_{3 \not\sim 7}$ + 2 d
pol 53_{obj} -1 +
pol $4_{2 \not\sim 4}$ $20_{2 \not\sim 9}$ + $22_{4 \not\sim 9}$ + 2 d
pol 53_{obj} -3 + -1 +
pol $9_{5 \not\sim 6}$ $26_{5 \not\sim 10}$ + $27_{6 \not\sim 10}$ + 2 d
pol 53_{obj} -5 + -3 + -1 +
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof

Verifying This Proof (For Real, This Time)

```
$ veripb --trace clique.opb clique-attempt-two.veripb
=== begin trace ===
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: soli x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: * bound, colour classes [ x1 x6 x9 ]
line 006: pol 7 19 + 24 + 2 d
  ConstraintId 044: 1 ~x1 1 ~x6 1 ~x9 >= 2
line 007: pol 42 43 +
  ConstraintId 045: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x8 1 x9 1 x10 1 x11 >= 3
...
  ConstraintId 061: 1 ~x5 1 ~x6 1 ~x10 >= 2
line 028: pol 53 57 + 59 + 61 +
  ConstraintId 062: 1 x8 1 x11 1 x12 >= 2
line 029: rup >= 1 ;
  ConstraintId 063: >= 1
line 030: output NONE
line 031: conclusion BOUNDS -4 -4
line 032: end pseudo-Boolean proof
=== end trace ===

Verification succeeded.
```

Different Clique Algorithms

Different search orders?

- ✓ Irrelevant for proof logging

Using local search to initialise?

- ✓ Just log the incumbent

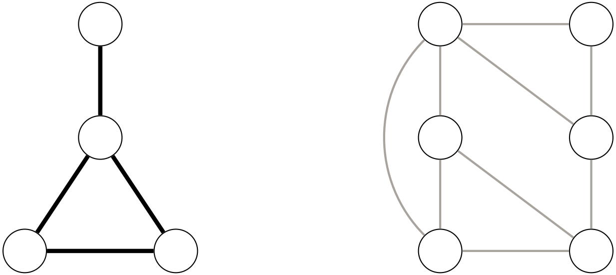
Different bound functions?

- Is cutting planes strong enough to justify every useful bound function ever invented?
- So far, seems like it…

Weighted cliques?

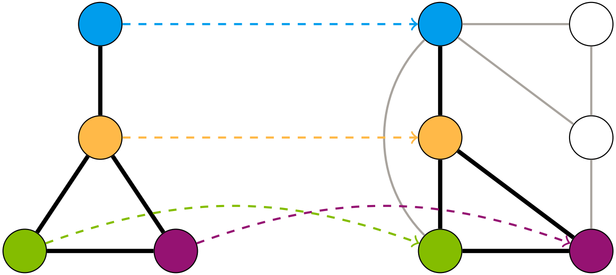
- ✓ Multiply a colour class by its largest weight
- ✓ Also works for vertices "split between colour classes"

Subgraph Isomorphism



- Find the pattern inside the target
- Applications in compilers, biochemistry, model checking, pattern recognition, …
- Often want to find all matches

Subgraph Isomorphism



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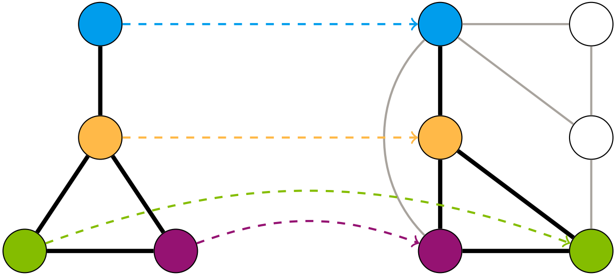

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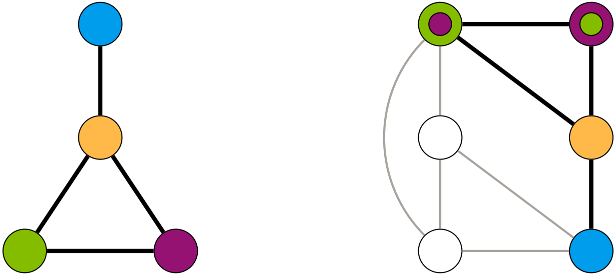

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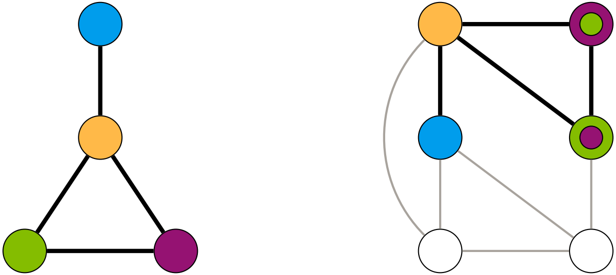

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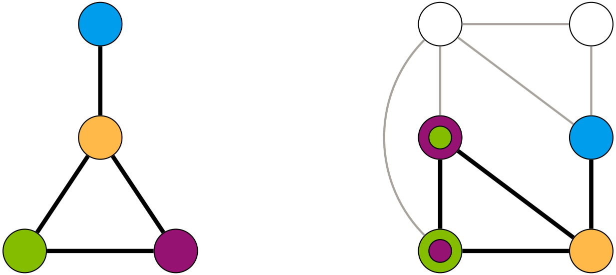

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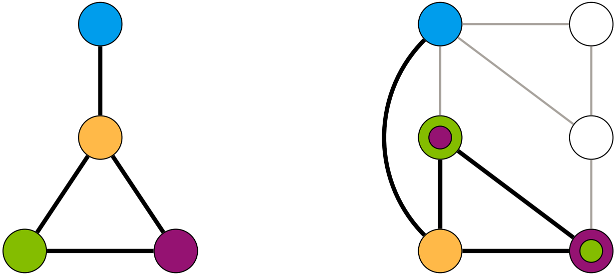

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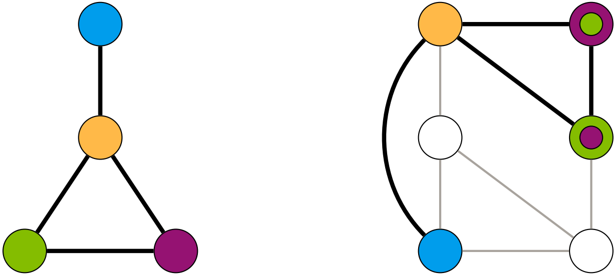

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Subgraph Isomorphism in Pseudo-Boolean Form

Each pattern vertex gets a target vertex:

$$\sum_{t \in \mathrm{V}(T)} x_{p,t} = 1 \qquad\qquad p \in \mathrm{V}(P)$$

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$$\sum_{p \in \mathrm{V}(P)} -x_{p,t} \geq -1 \qquad\qquad t \in \mathrm{V}(T)$$

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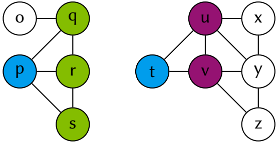
$$\sum_{p \in \mathrm{V}(P)} -x_{p,t} \geq -1 \qquad\qquad t \in \mathrm{V}(T)$$

Adjacency constraints, if p is mapped to t, then p's neighbours must be mapped to t's neighbours:

$$\overline{x}_{p,t} + \sum_{u \in \mathrm{N}(t)} x_{q,u} \geq 1 \qquad\qquad p \in \mathrm{V}(P),\ q \in \mathrm{N}(p),\ t \in \mathrm{V}(T)$$

Degree Reasoning in Cutting Planes

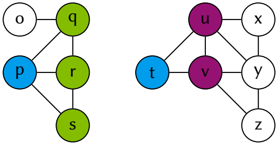


Pattern vertex p of degree $\deg(p)$ can never be mapped to target vertex t of degree $< \deg(p)$ in any subgraph isomorphism

Observe $N(p) = \{q, r, s\}$ and $N(t) = \{u, v\}$

We wish to derive $\overline{x}_{p,t} \geq 1$

Degree Reasoning in Cutting Planes



Adjacency:

$$\overline{x}_{p,t} + x_{q,u} + x_{q,v} \geq 1$$
$$\overline{x}_{p,t} + x_{r,u} + x_{r,v} \geq 1$$
$$\overline{x}_{p,t} + x_{s,u} + x_{s,v} \geq 1$$

Injectivity:

$$-x_{o,u} + -x_{p,u} + -x_{q,u} + -x_{r,u} + -x_{s,u} \geq -1$$
$$-x_{o,v} + -x_{p,v} + -x_{q,v} + -x_{r,v} + -x_{s,v} \geq -1$$

Literal axioms:

$$x_{o,u} \geq 0$$
$$x_{o,v} \geq 0$$
$$x_{p,u} \geq 0$$
$$x_{p,v} \geq 0$$

Add these together …

$$3 \cdot \overline{x}_{p,t} \geq 1$$

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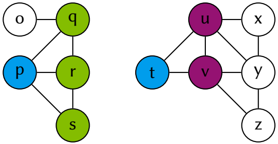

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$$x_{o,u} \geq 0$$
$$x_{o,v} \geq 0$$
$$x_{p,u} \geq 0$$
$$x_{p,v} \geq 0$$

Add these together and divide by 3 to get

$$\overline{x}_{p,t} \geq 1$$

Degree Reasoning in VeriPB

```
pol 18_{p~t:q} 19_{p~t:r} + 20_{p~t:s} +    * sum adjacency constraints
  12_{inj(u)} + 13_{inj(v)} +                * sum injectivity constraints
  xo_u + xo_v +                              * cancel stray xo_*
  xp_u + xp_v +                              * cancel stray xp_*
  3 d                                        * divide, and we're done
```

Or we can ask VeriPB to do the last bit of simplification automatically:

```
pol 18_{p~t:q} 19_{p~t:r} + 20_{p~t:s} +    * sum adjacency constraints
  12_{inj(u)} + 13_{inj(v)} +                * sum injectivity constraints
ia -1 : 1 ~xp_t >= 1 ;                       * desired conclusion is implied
```

Other Forms of Reasoning

We can also log all of the other things state of the art subgraph solvers do:

- Injectivity reasoning and filtering
- Distance filtering
- Neighbourhood degree sequences
- Path filtering
- Supplemental graphs

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- Neighbourhood degree sequences
- Path filtering
- Supplemental graphs

Proof steps are "efficient" using cutting planes

- Length of proof $\approx$ time complexity of the reasoning algorithms
- Most proof steps require only trivial additional computations

Limitations

Why trust the encoding?

- Correctness of encoding can be formally verified! Work in progress…

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Works up to moderately-sized hard instances

- Even an $O(n^3)$ encoding is painful
- Particularly bad when the pseudo-Boolean encoding talks about "non-edges" but large sparse graphs are "easy"

Code for Proof Logging Subgraph Solver

`https://github.com/ciaranm/glasgow-subgraph-solver`

Released under MIT Licence

What About Constraint Programming?

Non-Boolean variables?

Constraints?

- Encoding constraints in pseudo-Boolean form?
- Justifying inferences?

Reformulations?

Compiling CP Variables (1/2)

Given $A \in \{-3 \ldots 9\}$, the direct encoding is:

$$\begin{aligned} a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} \\ + a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} = 1 \end{aligned}$$

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This doesn't work for large domains...

We could use a binary encoding:

$$-16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq -3 \qquad \text{and}$$
$$16a_{\text{neg}} + -1a_{\text{b0}} + -2a_{\text{b1}} + -4a_{\text{b2}} + -8a_{\text{b3}} \geq -9$$

This doesn't propagate much, but that isn't a problem for proof logging

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Convention in what follows:

- Upper-case A, B, C are CP variables;
- Lower-case a, b, c are corresponding Boolean variables in PB encoding

Compiling CP Variables (2/2)

We can mix binary and an order encoding! Where needed, define:

$$a_{\geq 4} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 4$$
$$a_{\geq 5} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 5$$
$$a_{=4} \Leftrightarrow a_{\geq 4} \land \overline{a}_{\geq 5}$$

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$$a_{\geq 5} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 5$$
$$a_{=4} \Leftrightarrow a_{\geq 4} \wedge \overline{a}_{\geq 5}$$

When creating $a_{\geq i}$, also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values $j < i < h$ that already exist

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We can do this:

- Inside the pseudo-Boolean model, where needed
- Otherwise lazily during proof logging

Compiling Constraints

- Also need to compile every constraint to pseudo-Boolean form
- Doesn't need to be a propagating encoding
- Can use additional variables

Compiling Linear Inequalities

Given inequality

$$2A + 3B + 4C \geq 42$$

where $A, B, C \in \{-3 \ldots 9\}$

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$$2A + 3B + 4C \geq 42$$

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Encode in pseudo-Boolean form as

$$\begin{aligned} &-32a_{\text{neg}} + 2a_{\text{b0}} + 4a_{\text{b1}} + 8a_{\text{b2}} + 16a_{\text{b3}} \\ &+ -48b_{\text{neg}} + 3b_{\text{b0}} + 6b_{\text{b1}} + 12b_{\text{b2}} + 24b_{\text{b3}} \\ &+ -64c_{\text{neg}} + 4c_{\text{b0}} + 8c_{\text{b1}} + 16c_{\text{b2}} + 32c_{\text{b3}} \geq 42 \end{aligned}$$

Compiling Table Constraints

Constraints can be specified extensionally as list of feasible tuples, called a table
Variable assignments must match some row in table

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Variable assignments must match some row in table

Given table constraint

$$(A, B, C) \in [(1, 2, 3), (1, 3, 4), (2, 2, 5)]$$

define

$$3\bar{t}_1 + a_{=1} + b_{=2} + c_{=3} \geq 3 \qquad \text{i.e.,} \quad t_1 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3})$$
$$3\bar{t}_2 + a_{=1} + b_{=4} + c_{=4} \geq 3 \qquad \text{i.e.,} \quad t_2 \Rightarrow (a_{=1} \wedge b_{=4} \wedge c_{=4})$$
$$3\bar{t}_3 + a_{=2} + b_{=2} + c_{=5} \geq 3 \qquad \text{i.e.,} \quad t_3 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5})$$

using tuple selector variables

$$t_1 + t_2 + t_3 = 1$$

Encoding Constraint Definitions

Already know how to do it for any constraint with a sane encoding using some combination of

- CNF
- Integer linear inequalities
- Table constraints
- Auxiliary variables

Simplicity is important, propagation strength isn’t

Justifying Search

Mostly this works as in earlier examples

Restarts are easy

No need to justify guesses or decisions — only justify backtracking

Justifying Inference

Key idea

Anything the constraint programming solver knows must follow from unit propagation of guessed assignments on constraints in proof log

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If it follows from unit propagation on the encoding, nothing needed

Some propagators and encodings need RUP steps for inferences

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Anything the constraint programming solver knows must follow from unit propagation of guessed assignments on constraints in proof log

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Some propagators and encodings need RUP steps for inferences

- A lot of propagators are effectively "doing a little bit of lookahead" but in an efficient way

A few need explicit cutting planes justifications written to the proof log

- Linear inequalities just need to multiply and add
- All-different needs a bit more

Justifying All-Different Failures

$$
\begin{array}{rl}
V \in & \{\, 1 \quad \quad \quad 4 \quad 5 \,\} \\
W \in & \{\, 1 \quad 2 \quad 3 \quad \quad \,\} \\
X \in & \{\, \quad 2 \quad 3 \quad \quad \,\} \\
Y \in & \{\, 1 \quad \quad 3 \quad \quad \,\} \\
Z \in & \{\, 1 \quad \quad 3 \quad \quad \,\}
\end{array}
$$

Justifying All-Different Failures

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$$W \in \{1 \quad 2 \quad 3 \quad \quad \}$$
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$$Y \in \{1 \quad \quad 3 \quad \quad \}$$
$$Z \in \{1 \quad \quad 3 \quad \quad \}$$

Justifying All-Different Failures

$V \in \{1 \quad 4 \quad 5\}$

$W \in \{1 \quad 2 \quad 3\}$ $\quad w_{=1} + w_{=2} + w_{=3} \geq 1$ $\quad$ [W takes some value]

$X \in \{2 \quad 3\}$

$Y \in \{1 \quad 3\}$

$Z \in \{1 \quad 3\}$

Justifying All-Different Failures

$$V \in \{1, 4, 5\}$$

$$W \in \{1, 2, 3\} \qquad w_{=1} + w_{=2} + w_{=3} \geq 1 \quad [\text{ } W \text{ takes some value }]$$

$$X \in \{2, 3\} \qquad x_{=2} + x_{=3} \geq 1 \quad [\text{ } X \text{ takes some value }]$$

$$Y \in \{1, 3\} \qquad y_{=1} + y_{=3} \geq 1 \quad [\text{ } Y \text{ takes some value }]$$

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$$V \in \{1 \quad 4 \quad 5\}$$

$W \in \{1\ 2\ 3\}$	$w_{=1} + w_{=2} + w_{=3}$	≥ 1	[W takes some value]
$X \in \{2\ 3\}$	$x_{=2} + x_{=3}$	≥ 1	[X takes some value]
$Y \in \{1\ 3\}$	$y_{=1} + y_{=3}$	≥ 1	[Y takes some value]
$Z \in \{1\ 3\}$	$z_{=1} + z_{=3}$	≥ 1	[Z takes some value]
$\rightarrow$	$-v_{=1} + -w_{=1} + -y_{=1} + -z_{=1}$	≥ -1	[At most one variable = 1]
$\rightarrow$	$-w_{=2} + -x_{=2}$	≥ -1	[At most one variable = 2]
$\rightarrow$	$-w_{=3} + -x_{=3} + -y_{=3} + -z_{=3}$	≥ -1	[At most one variable = 3]

Justifying All-Different Failures

$V \in \{1 \quad \quad 4 \quad 5\}$

$W \in \{1\ 2\ 3\}$	$w_{=1} + w_{=2} + w_{=3}$	≥ 1	[W takes some value]
$X \in \{2\ 3\}$	$x_{=2} + x_{=3}$	≥ 1	[X takes some value]
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$\rightarrow$	$-w_{=3} + -x_{=3} + -y_{=3} + -z_{=3}$	≥ -1	[At most one variable = 3]
	$-v_{=1}$	≥ 1	[Sum all constraints so far]

Justifying All-Different Failures

$$
\begin{array}{rlll}
V \in \{1, 4, 5\} & & & \\
W \in \{1, 2, 3\} & w_{=1} + w_{=2} + w_{=3} & \geq 1 & \text{[} W \text{ takes some value]} \\
X \in \{2, 3\} & x_{=2} + x_{=3} & \geq 1 & \text{[} X \text{ takes some value]} \\
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Z \in \{1, 3\} & z_{=1} + z_{=3} & \geq 1 & \text{[} Z \text{ takes some value]} \\
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 & -v_{=1} & \geq 1 & \text{[Sum all constraints so far]} \\
 & v_{=1} & \geq 0 & \text{[Variable } v_{=1} \text{ non-negative]}
\end{array}
$$

Justifying All-Different Failures

$V \in \{1 \quad 4 \quad 5\}$			
$W \in \{1 \quad 2 \quad 3\}$	$w_{=1} + w_{=2} + w_{=3}$	≥ 1	[W takes some value]
$X \in \{2 \quad 3\}$	$x_{=2} + x_{=3}$	≥ 1	[X takes some value]
$Y \in \{1 \quad 3\}$	$y_{=1} + y_{=3}$	≥ 1	[Y takes some value]
$Z \in \{1 \quad 3\}$	$z_{=1} + z_{=3}$	≥ 1	[Z takes some value]
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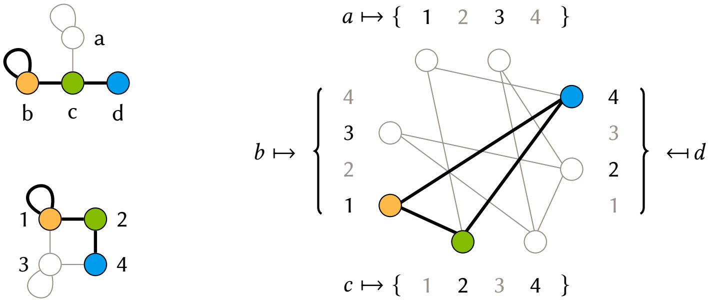
$\rightarrow$	$-w_{=3} + -x_{=3} + -y_{=3} + -z_{=3}$	≥ -1	[At most one variable = 3]
	$-v_{=1}$	≥ 1	[Sum all constraints so far]
	$v_{=1}$	≥ 0	[Variable $v_{=1}$ non-negative]
	0	≥ 1	[Sum above two constraints]

Reformulation

Auto-tabulation is possible

- Heavy use of extension variables

Can re-encode maximum common subgraph as a clique problem, without changing pseudo-Boolean encoding



High Level Modelling Languages?

High level modelling languages like MiniZinc and Essence have complicated compilers

How do we know we're giving a proof for the problem the user actually specified?

This would need a modelling language with formally specified semantics...

Code

https://github.com/ciaranm/glasgow-constraint-solver

Released under MIT Licence

Supports proof logging for global constraints including:

- All-different
- Integer linear inequality (including for very large domains)
- Smart table and regular
- Minimum / maximum of an array
- Element
- Absolute value
- (Hamiltonian) Circuit

Details in [EGMN20, GMN22, MM23]

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When is it allowed to derive a new constraint? If it is (clear that it is) implied?

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Sometimes weaker criterion needed — recall that to get variable a encoding

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we introduced pseudo-Boolean constraints

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Cutting planes method inherently cannot certify such constraints — they are not implied!

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Cutting planes method inherently cannot certify such constraints — they are not implied!

Wish to allow without-loss-of-generality arguments that can derive non-implied constraints

Strengthening Rules (and Symmetry)

VeriPB supports different forms of strengthening rules that enable such w.l.o.g. arguments

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Care is needed in combination with deletion

Can be very powerful: VeriPB can certify automatic symmetry breaking for SAT

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in *DRAT-Trim* [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM+23])*

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- Mixed integer linear programming *(some work on SCIP in [CGS17, EG21])*
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- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas

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And more...

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- Talk to us if you want to join the proof logging revolution! ☺
  We're happy to collaborate, and we're hiring

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity

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The end.

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The end. Or rather, the beginning!

References for Getting Started with VeriPB

https://gitlab.com/MIAOresearch/software/VeriPB

Released under MIT Licence

Various features to help development:

- Extended variable name syntax allowing human-readable names
- Proof tracing
- "Trust me" assertions for incremental proof logging

Documentation:

- Description of VeriPB checker [BMM+23] used in SAT 2023 competition (https://satcompetition.github.io/2023/checkers.html)
- Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23]
- Lots of concrete example files at https://gitlab.com/MIAOresearch/software/VeriPB

References I

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Parity Reasoning: Experimental Evaluation

Implemented parity reasoning and PB proof logging engine[2]

Also DRAT proof logging for XOR constraints as described in [PR16]

Experiments with MiniSat[3]

Set-up:[4]

- Intel Core i5-1145G7 @2.60GHz $\times$ 4
- Memory limit 8GiB
- Disk write speed roughly 200 MiB/s
- Read speed of 2 GiB/s

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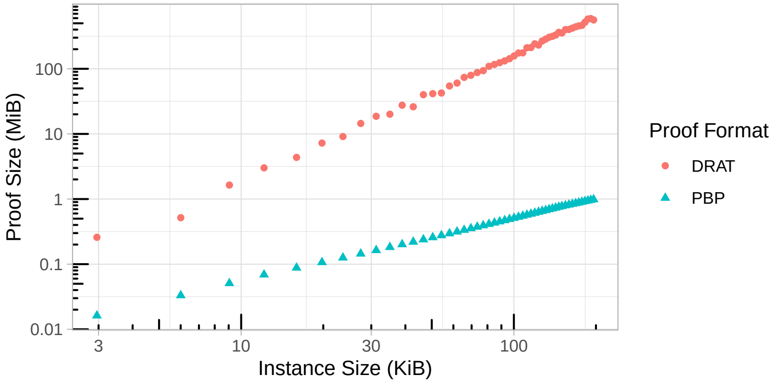
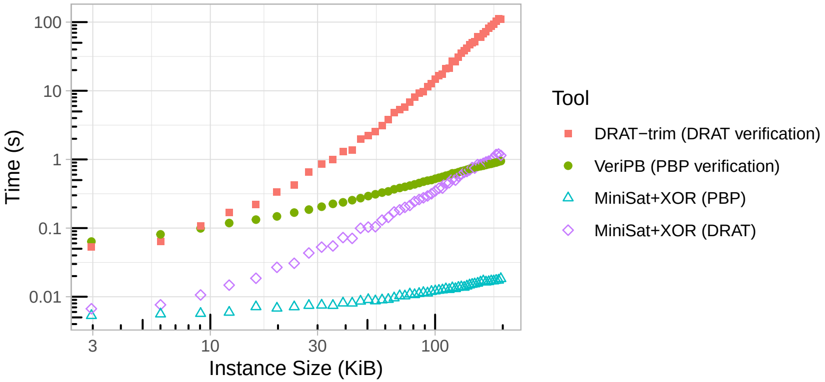
[2] https://gitlab.com/MIAOresearch/tools-and-utlities/xorengine
[3] http://minisat.se/
[4] Tools, benchmarks, data and evaluation scripts available at https://doi.org/10.5281/zenodo.7083485

Parity Reasoning: Proof Size for DRAT and PB Proof Logging



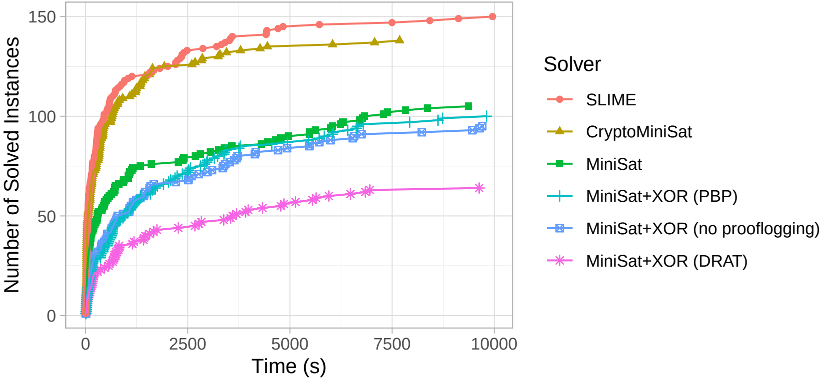
Proof sizes for Tseitin formulas using DRAT and pseudo-Boolean proof logging

Parity Reasoning: Solving and Proof Checking Time



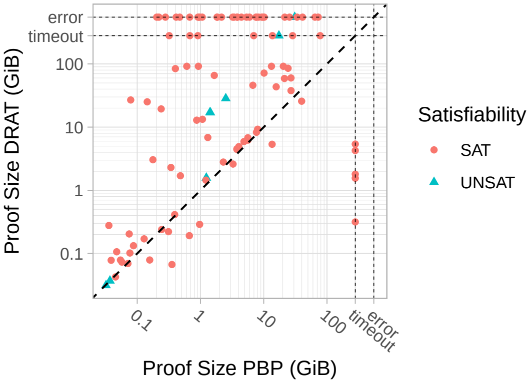
Solving and proof checking time for Tseitin formulas using DRAT and PB proof logging

Parity Reasoning: Crypto Track of SAT 2021 Competition



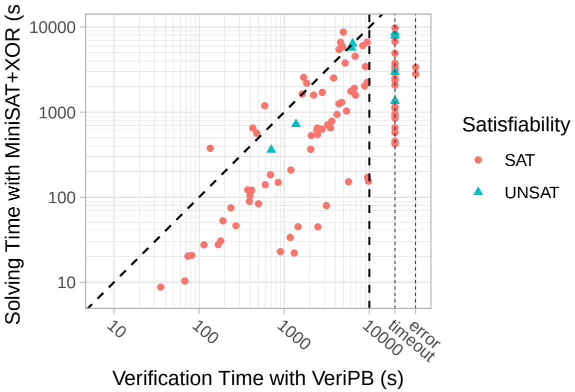
Cumulative plot for the crypto track of the SAT Competition 2021

Parity Reasoning: Crypto Track Proof Size



DRAT and PB proof sizes for crypto track of SAT Competition 2021

Parity Reasoning: Crypto Track Solving & Proof Checking Time



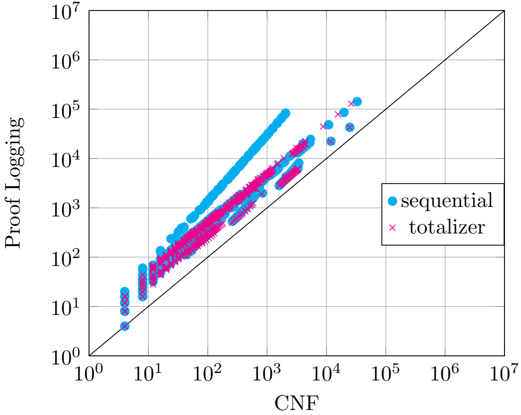
Time required for solving and proof checking for cryptographic instances

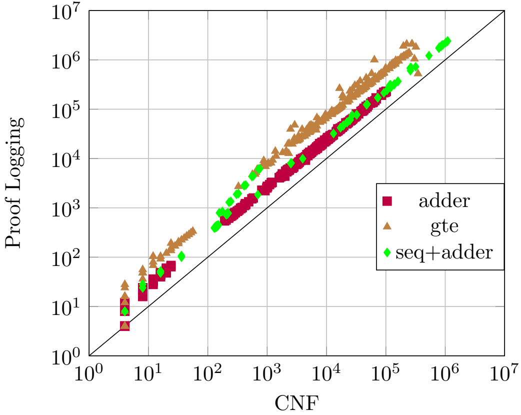
PB-to-CNF Translation: Experimental Evaluation

- Certified translations for CNF encodings with *VeritasPBLib*[5]
  - Sequential counter [Sin05]
  - Totalizer [BB03]
  - Generalized totalizer [JMM15]
  - Adder network [ES06]
- Proofs verified by proof checker VeriPB
- Formulas solved with fork of Kissat[6] syntactically modified to output VeriPB proofs
- Benchmarks from PB 2016 Evaluation[7] in 3 categories
  - Only cardinality constraints (sequential counter, totalizer)
  - Only general 0-1 ILP constraints (generalized totalizer, adder network)
  - Mixed cardinality & general 0-1 ILP constraints (sequential counter + adder network)

[5] https://github.com/forge-lab/VeritasPBLib
[6] https://gitlab.com/MIAOresearch/tools-and-utilities/kissat_fork
[7] http://www.cril.univ-artois.fr/PB16/

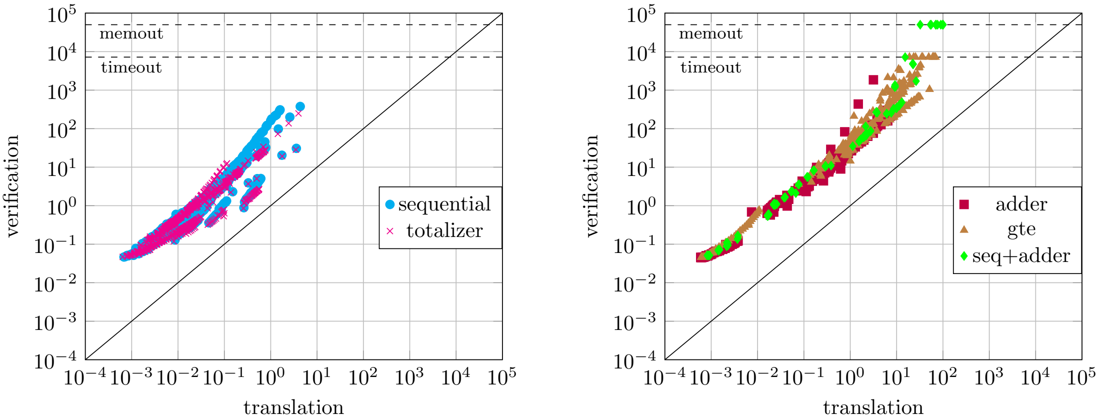
PB-to-CNF: CNF Size vs Proof Size in KiB





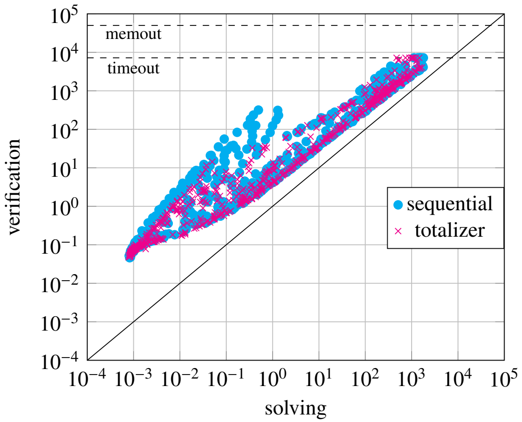
- Nice scaling for proof size in terms of original CNF formula size
- Except for some sequential encoding cases (which is not such a great encoding anyway)

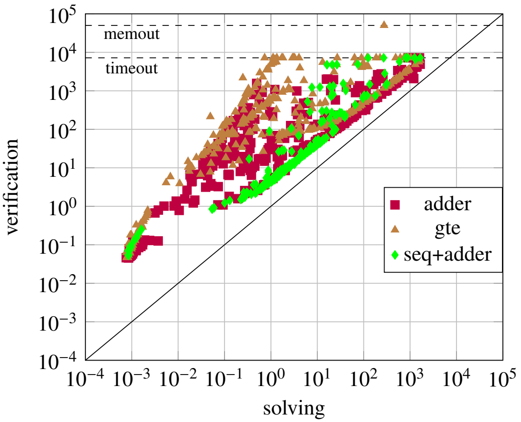
PB-to-CNF: Translation Time vs Proof Checking Time in Seconds



- Translation faster — only has to generate clauses and proof
- Proof checking slower — has to verify full proof

PB-to-CNF: Solving Time vs Proof Checking Time in Seconds





- Room for improvement of end-to-end proof checking process
- But even first proof-of-concept implementation shows our approach is viable

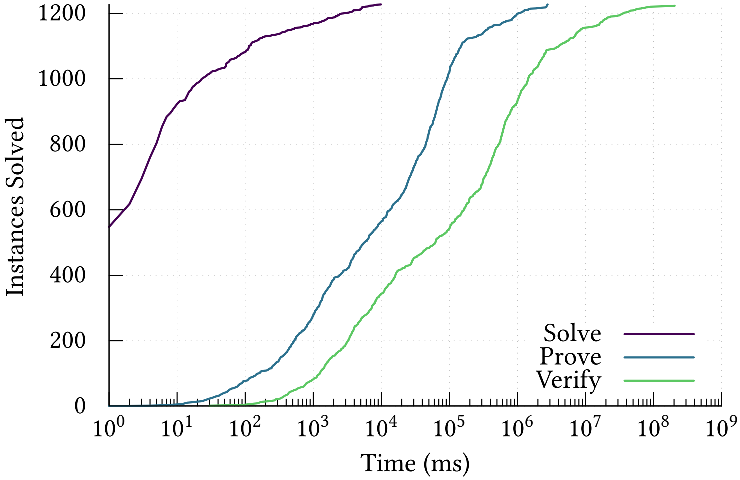
Clique Solving: Experimental Evaluation

- Implemented in the *Glasgow Subgraph Solver*
  - Bit-parallel, can perform a colouring and recursive call in under a microsecond
- 59 of the 80 DIMACS instances take under 1,000 seconds to solve without logging
- Produced and verified proofs for 57 of these 59 instances (the other two reached 1TByte disk space)
- Mean slowdown from proof logging is 80.1 (due to disk I/O)
- Mean verification slowdown a further 10.1
- Approximate implementation effort: one Masters student

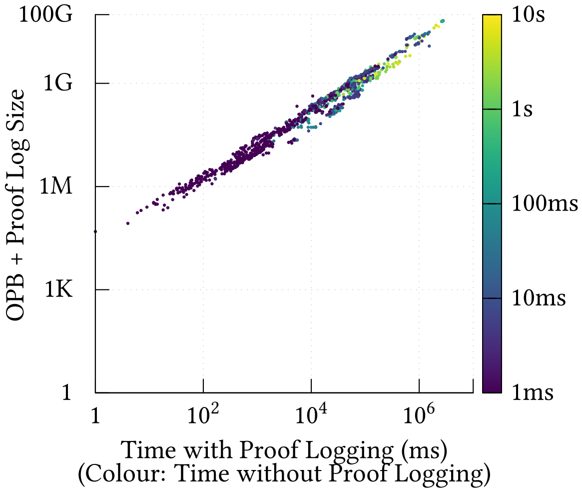
Subgraph Isomorphism Solving: Experimental Evaluation (1/3)

- The Pseudo-Boolean models can be large: had to restrict to instances with no more than 260 vertices in the target graph
- Took enumeration instances which could be solved without proof logging in under ten seconds
- 1,227 instances from Solnon's benchmark collection:
  - 789 unsatisfiable, up to 50,635,140 solutions in the rest
  - 498 instances solved without guessing
  - Hardest solved satisfiable and unsatisfiable instances required 53,605,482 and 2,074,386 recursive calls

Subgraph Isomorphism Solving: Experimental Evaluation (2/3)



Subgraph Isomorphism Solving: Experimental Evaluation (3/3)



Constraint Programming: How Expensive is Proof Logging? (1/2)

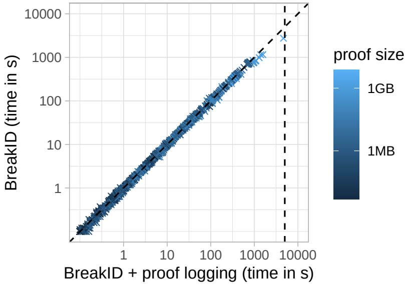
- Laurent D. Michel, Pierre Schaus, Pascal Van Hentenryck: *MiniCP: A Lightweight Solver for Constraint Programming* [MSH21]
- Five benchmark problems allowing comparison of solvers “doing the same thing”:
  - Simple models
  - Fixed search order and well-defined propagation consistency levels
  - Few global constraints
- Probably close to the worst case for proof logging performance
- Also: Crystal Maze and World’s Hardest Sudoku

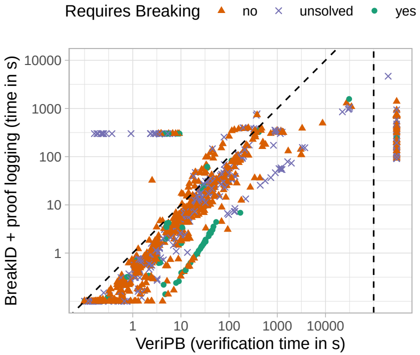
Constraint Programming: How Expensive is Proof Logging? (2/2)

- Our solver: faster than the fastest of *MiniCP*, *OscaR*, and *Choco*
- Proof logging slowdown: between 8.4 and 61.1 factor
  - 800,000 to 3,000,000 inferences per second
  - Proof logs can be hundreds of GBytes
  - No effort put into making the proof-writing code run fast
- Verification slowdown: a further factor 10 to 100
  - Probably possible to reduce this substantially if we are prepared to put more care into writing proofs

SAT Symmetry Breaking: Experimental Evaluation

- Evaluated on SAT competition benchmarks
- *BreakID* [DBBD16, Bre] used to find and break symmetries





- Proof logging overhead negligible
- Proof checking at most 20 times slower than solving for 95% of instances