On Symmetries and Certification

Bart Bogaerts
(Thanks to co-conspirators Jo Devriendt, Stephan Gocht, Ciaran McCreesh, Jakob Nordström)
Vrije Universiteit Brussel

Dagstuhl Seminar 22411
Theory and Practice of SAT and Combinatorial Solving

ARTIFICIAL
INTELLIGENCE
RESEARCH GROUP

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  - σ respects negation: $\overline{\sigma(x)} = \sigma(\overline{x})$
  - $F\upharpoonright_\sigma = F$
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Consider the formula F:

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$$\overline{c} \vee \overline{d} \vee x \vee y \qquad d \vee \overline{a} \vee x \vee y$$

The permutation

$$(ab\overline{c}d)(xy)(\overline{a}\overline{b}c\overline{d})(\overline{x}\overline{y})$$

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is a symmetry of F since $F\restriction_\sigma$ is

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- Symmetric problems are often problematic for vanilla CDCL solvers (insert obligatory reference to PH principle here)

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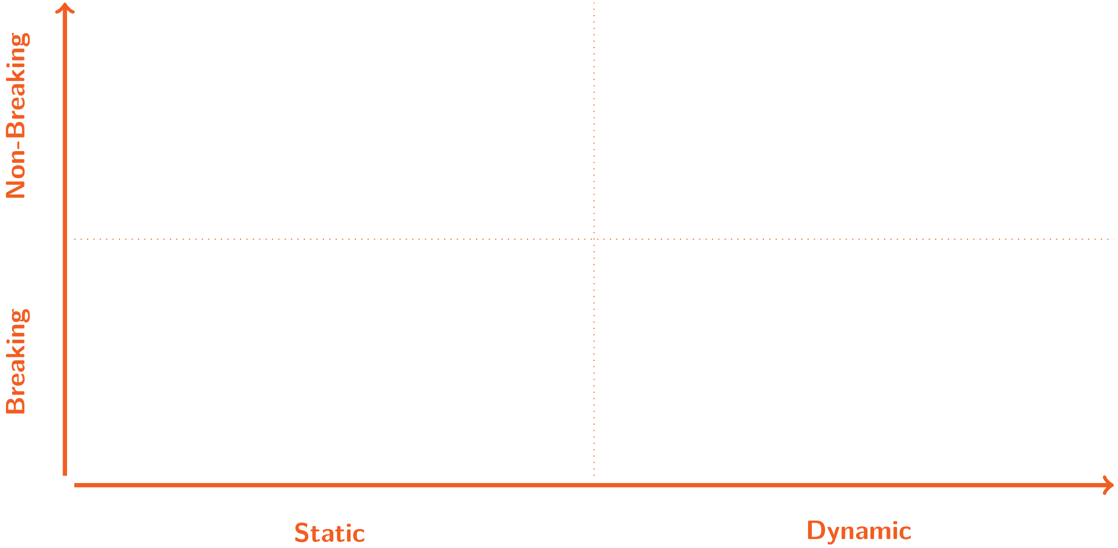
OUTLINE OF THIS TALK

SYMMETRY HANDLING TECHNIQUES



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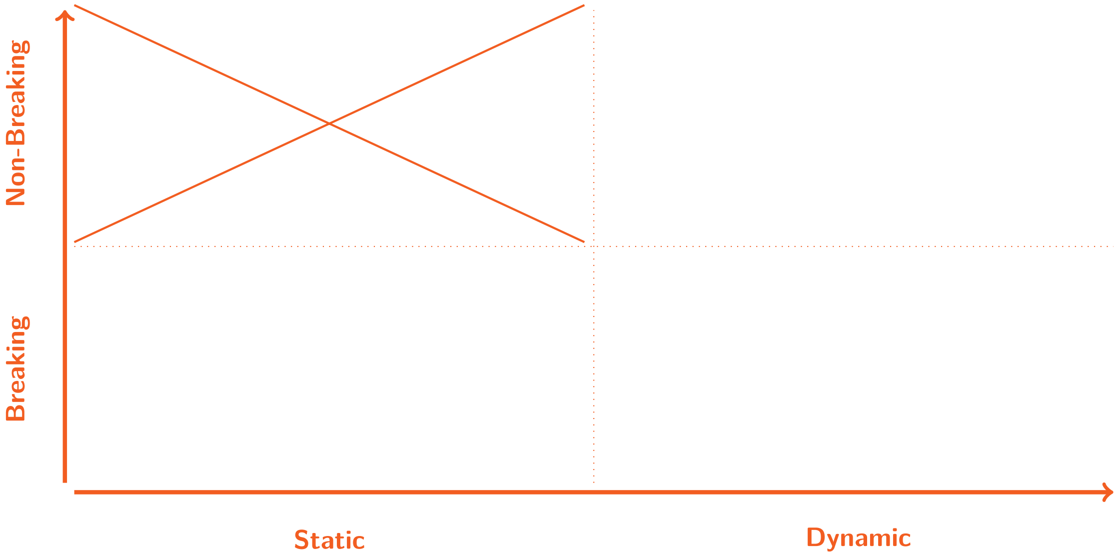

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

Add lex-leader constraint
for symmetries of F:

set of clauses B such that
$\alpha \models B$ iff $\alpha \preceq_{lex} \alpha \circ \sigma$

Breaking

Global symmetry breaking
SHATTER [ASM06]
BREAKID [DBBD16]

Static

Dynamic

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

Add lex-leader constraint
for symmetries of subformulas of F:

Breaking

Global symmetry breaking
SHATTER [ASM06]
BREAKID [DBBD16]

Local symmetry breaking [BS07]

Static

Dynamic

SYMMETRY HANDLING TECHNIQUES

	Static	Dynamic
Non-Breaking	For "simple symmetries", instead of branching on variables, on the number of variables that are true	
Breaking	Global symmetry breaking SHATTER [ASM06] BREAKID [DBBD16]	Local symmetry breaking [BS07] Asymmetric branching SYMCHAFF [Sab09]

SYMMETRY HANDLING TECHNIQUES

	Static	Dynamic
Non-Breaking	Add lex-leader constraint for symmetries of F when these clauses would propagate	
Breaking	Global symmetry breaking SHATTER [ASM06] BREAKID [DBBD16]	Local symmetry breaking [BS07] Asymmetric branching SYMCHAFF [Sab09] Effective symmetry breaking [MBCK18]

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

Propagator for $\preceq_{lex}$-minimality (graphs)

Breaking

Global symmetry breaking
SHATTER [ASM06]
BREAKID [DBBD16]

Local symmetry breaking [BS07]
Asymmetric branching SYMCHAFF [Sab09]
Effective symmetry breaking [MBCK18]
Sat modulo symmetries [KS21]

Static | **Dynamic**

SYMMETRY HANDLING TECHNIQUES

	Static	Dynamic
Non-Breaking	When SAT solver learns c, also learn $c\restriction_\sigma$ (if this seems "interesting")	Symmetric Learning [HKM⁺05] [SHvM09, BNOS10, DBD⁺12, DBB17]
Breaking	Global symmetry breaking SHATTER [ASM06] BREAKID [DBBD16]	Local symmetry breaking [BS07] Asymmetric branching SYMCHAFF [Sab09] Effective symmetry breaking [MBCK18] Sat modulo symmetries [KS21]

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

Hybrid combination of
Effective symmetry breaking predicates
(first)
and symmetric learning
(for symmetries not broken completely)

Symmetric Learning [HKM+05]
[SHvM09, BNOS10, DBD+12, DBB17]

ESBP+SP [MBK19]

Breaking

Global symmetry breaking
Shatter [ASM06]
BreakID [DBBD16]

Local symmetry breaking [BS07]
Asymmetric branching SymChaff [Sab09]
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Static **Dynamic**

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Symmetric learning

- Recently proposed proof logging [TD20]
  1. Special-purpose, specific approach
  2. Requires adding explicit concept of symmetries
  3. Not compatible with preprocessing techniques

  Better to keep proof system super-simple(?)

THE VERIPB PROOF SYSTEM

A proof system for **pseudo-Boolean optimization problems**

- Reasons with general pseudo-Boolean constraints
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Details about the proof checker, see Stephan Gocht's PhD thesis [Goc22]

PSEUDO-BOOLEAN CONSTRAINTS

Pseudo-Boolean constraints are 0-1 integer linear constraints

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- as before, variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

PSEUDO-BOOLEAN REASONING: CUTTING PLANES [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

REDUNDANCE-BASED STRENGTHENING

- C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable
- Adding redundant constraints should be OK
- Notions such as RAT [JHB12] and propagation redundancy [HKB17]

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Redundance-based strengthening [BT19, GN21]

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$$

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Fact

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$
- Implication should be efficiently verifiable (which is the case, e.g., if all constraints in $(F \wedge C)\restriction_\omega$ are RUP)

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Deal with symmetries by switching focus to optimization

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Spoiler alert:
For decision problem, nothing stops us from inventing objective function (like lexicographic order $\sum_{i=1}^{n} 2^i \cdot x_i$)

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7. ...
8. Can't go on forever, so finally reach α' satisfying $F \wedge D$

STRENGTH OF DOMINANCE RULE

Dominance-based strengthening (stronger, still simplified) [BGMN22]

If $D_1, D_2, \ldots, D_{m-1}$ have been derived from F (maybe using dominance), then can derive also D_m if exists witness substitution ω such that

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Further extensions:

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- Switch between different orders in same proof

STRATEGY FOR SAT SYMMETRY BREAKING

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2. Derive pseudo-Boolean lex-leader constraint

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\begin{aligned}
C_\sigma &\doteq f \leq f\restriction_\sigma \\
&\doteq \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0
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3. Derive CNF encoding of lex-leader constraints from PB constraint (in same spirit as [GMNO22])

$$
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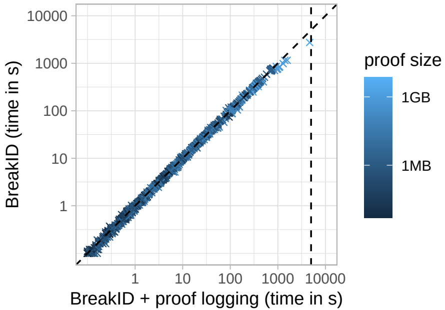
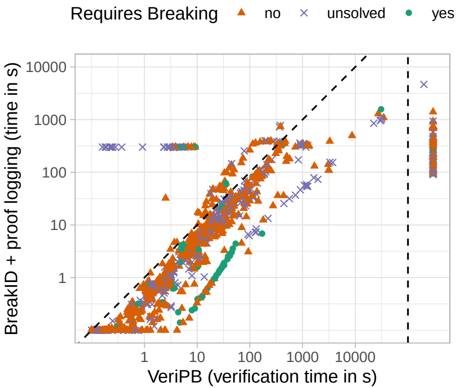
&\overline{y}_j \vee y_{j-1} && y_j \vee \overline{y}_{j-1} \vee \sigma(x_j)
\end{aligned}
$$

EXPERIMENTAL EVALUATION

- Evaluated on SAT competition benchmarks
- BreakID [DBBD16, Bre] used to find and break symmetries





- proof logging overhead negligible
- verification at most 20 times slower than solving for 95% of instances

SYMMETRY BREAKING: EXAMPLE

Example (Pigeonhole principle formula)

- Variables p_{ij} $(1 \leq i \leq 4, 1 \leq j \leq 3)$ true iff pigeon i in hole j
- Focus on pigeon symmetries — notation:
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Order: "Pigeon 1 preferred in the largest possible hole; next pigeon 2, ..."

$$f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^{9} \cdot p_{11} + 2^{8} \cdot p_{23} + \cdots + 1 \cdot p_{41}$$

BREAKING A SINGLE SIMPLE SYMMETRY (EXAMPLE)

- F is a formula expressing PHP constraints with $F\restriction_{\sigma_{(12)}} = F$
- Want to add constraint C_{12} breaking $\sigma_{(12)}$ — should be satisfied by α iff α "at least as good" as $\sigma_{(12)}(\alpha)$

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$$F \wedge f > f\restriction_{\sigma_{(12)}} \models F\restriction_{\sigma_{(12)}} \wedge f \geq f\restriction_{\sigma_{(12)}} \wedge f\restriction_{\sigma_{(12)}} \leq f$$

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Similar to DRAT symmetry breaking [HHW15]

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- Breaking two symmetries
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  If it is falsified, we can "restore" its truth by applying $\sigma_{(1234)}$ once, twice, or thrice.

BREAKING SYMMETRIES WITH THE DOMINANCE RULE (1/2)

Definition

Given a symmetry σ, the (pseudo-Boolean) breaking constraint of σ is

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Theorem

C_σ can be derived from F using dominance with witness σ

$$F \wedge \neg C_\sigma \models F\restriction_\sigma \wedge f\restriction_\sigma < f$$

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Breaking symmetries with the dominance rule

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Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

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Thank you for your attention!

STRATEGY FOR SAT SYMMETRY BREAKING

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (searching lexicographically smallest assignment satisfying formula)
2. Derive pseudo-Boolean lex-leader constraint

$$
\begin{aligned}
C_\sigma &\doteq f \leq f\restriction_\sigma \\
&\doteq \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0
\end{aligned}
$$

3. Derive CNF encoding of lex-leader constraints from PB constraint (in same spirit as [GMNO22])

$$
\begin{aligned}
&y_0 && \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j \\
&\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) && y_j \vee \overline{y}_{j-1} \vee \overline{x}_j \\
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Define y_j to be true if x_k equals $\sigma(x_k)$ for all $k \leq j$

$$y_k \Leftrightarrow y_{k-1} \wedge (x_k \Leftrightarrow \sigma(x_k))$$

(derivable with redundance rule)

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(derivable with redundance rule) If y_k is true, x_k is at most $\sigma(x_k)$
(derivable from the PB breaking constraint)

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

Pseudo-Boolean breaking constraint

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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y_0

Derivable by redundance with witness $\omega : y_0 \mapsto 1$

$$F \wedge D \wedge \{\overline{y}_0\} \models (F \wedge D)\restriction_\omega \wedge \{y_0\}\restriction_\omega$$

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Derivable by redundance with witness $\omega : y_1 \mapsto 0$

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Derivable by redundance with witness $\omega : y_1 \mapsto 0$ (same argument)

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$$F \wedge D \wedge \neg(y_1 \vee \overline{y}_0 \vee \overline{x}_1) \models (F \wedge D)\restriction_\omega \wedge \{y_1 \vee \overline{y}_0 \vee \overline{x}_1\}\restriction_\omega$$

$$F \wedge D \wedge \{\overline{y}_1 \wedge y_0 \wedge x_1) \models \dots \wedge D\restriction_\omega \wedge \dots$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$

$\overline{y}_1 \vee y_0$

$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$

$y_1 \vee \overline{y}_0 \vee \overline{x}_1$

$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$

Derivable by redundance with witness $\omega : y_1 \mapsto 1$ (same argument)

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$y_0$$

$$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee y_0$$

$$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$$

$$y_1 \vee \overline{y}_0 \vee \overline{x}_1$$

$$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee \overline{x}_2 \vee \sigma(x_2)$$

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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$$y_0$$

$$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee y_0$$

$$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$$

$$y_1 \vee \overline{y}_0 \vee \overline{x}_1$$

$$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee \overline{x}_2 \vee \sigma(x_2)$$

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$+ 2^{n-1} \cdot \left(\overline{y}_1 + \overline{\sigma(x_1)} + x_1 \geq 1\right)$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$y_0$$

$$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee y_0$$

$$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$$

$$y_1 \vee \overline{y}_0 \vee \overline{x}_1$$

$$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$$

$$\overline{y}_1 \vee \overline{x}_2 \vee \sigma(x_2)$$

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$+ 2^{n-1} \cdot \left(\overline{y}_1 + \overline{\sigma(x_1)} + x_1 \geq 1\right)$$

$$2^{n-1} \cdot \overline{y}_1 + \sum_{i=2}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

DETAILED DERIVATION OF CNF BREAKING CONSTRAINTS

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$

$\overline{y}_1 \vee y_0$

$\overline{y}_1 \vee \overline{\sigma(x_1)} \vee x_1$

$y_1 \vee \overline{y}_0 \vee \overline{x}_1$

$y_1 \vee \overline{y}_0 \vee \sigma(x_1)$

$\overline{y}_1 \vee \overline{x}_2 \vee \sigma(x_2)$

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$
$$+ 2^{n-1} \cdot \left(\overline{y}_1 + \overline{\sigma(x_1)} + x_1 \geq 1\right)$$

$$2^{n-1} \cdot \overline{y}_1 + \sum_{i=2}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

The clause to derive is RUP wrt this constraint

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