Certifying Pareto Optimality in Multi-Objective Maximum Satisfiability

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KU Leuven

Dagstuhl Seminar 25231

DTAI
Declarative Languages & Artificial Intelligence
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ARTIFICIAL INTELLIGENCE RESEARCH GROUP

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TAKE-AWAY MESSAGE

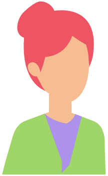
Proof logging for multi-objective problems is feasible [JBBJ25]

OUTLINE

1. Problem Setting
2. Background
3. Proofs for MO-MaxSAT
4. Conclusions

MULTI-OBJECTIVE OPTIMIZATION

HOW TO DEAL WITH CONFLICTING OBJECTIVES



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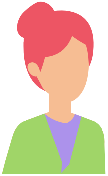

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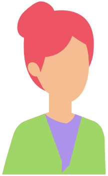

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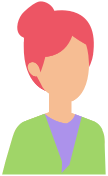

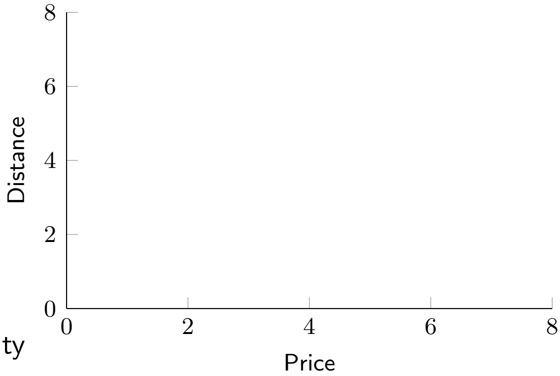

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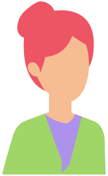

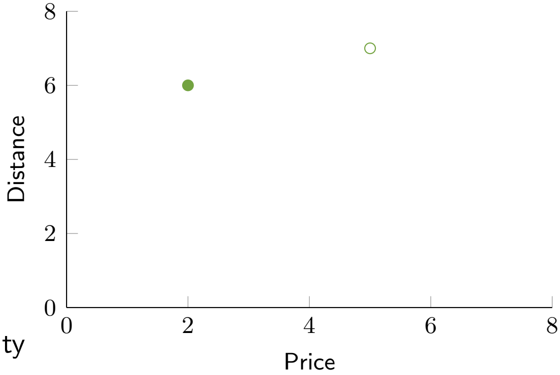

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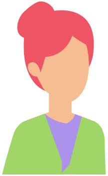

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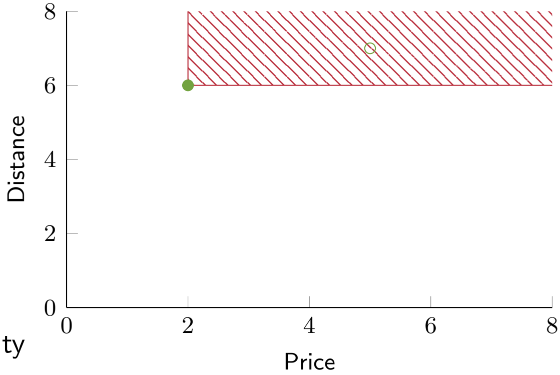

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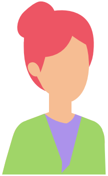

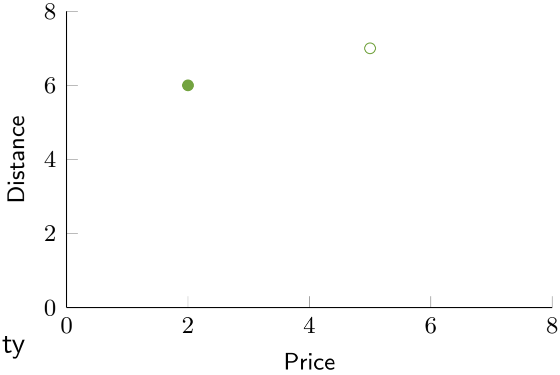

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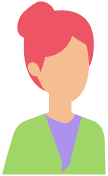

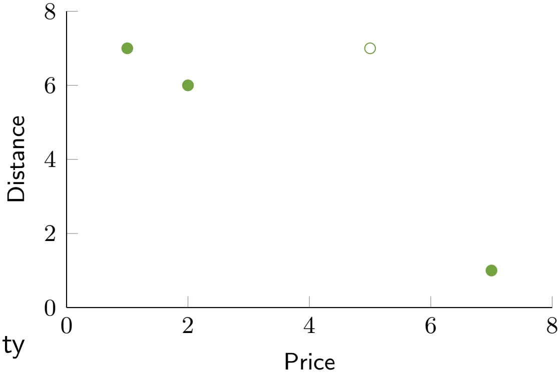

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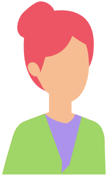

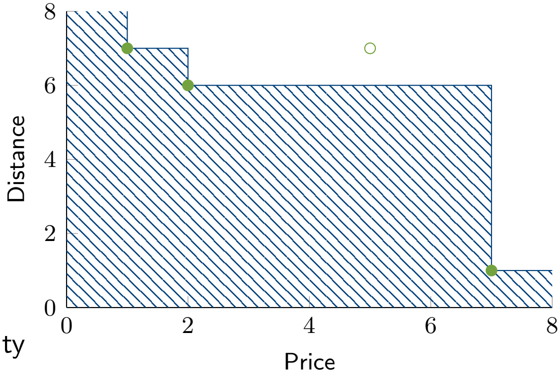

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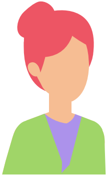
THE PROBLEM

CAN YOU TRUST MY SOLVER?



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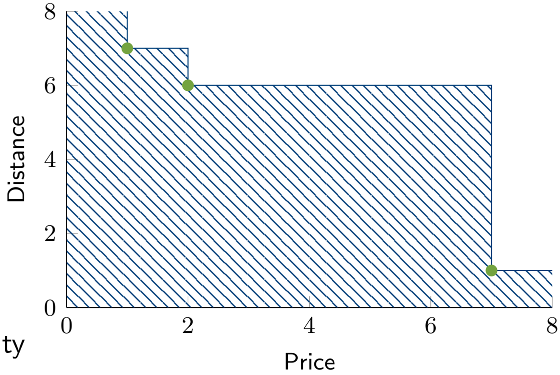

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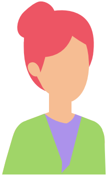

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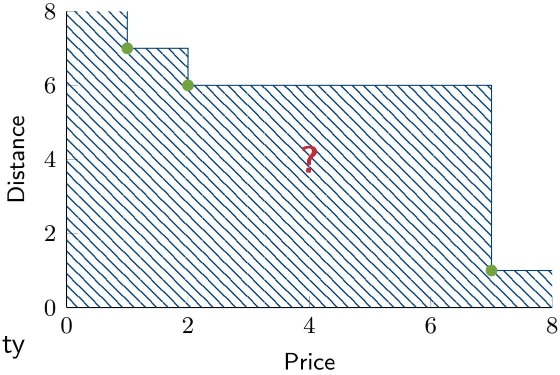

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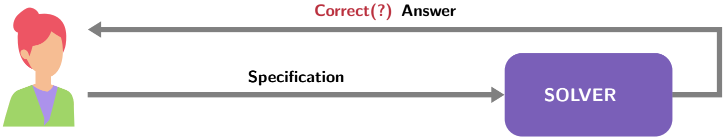

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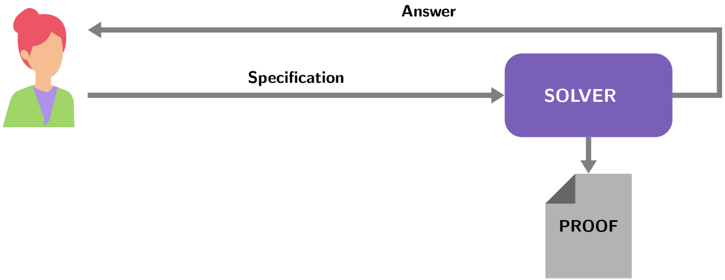

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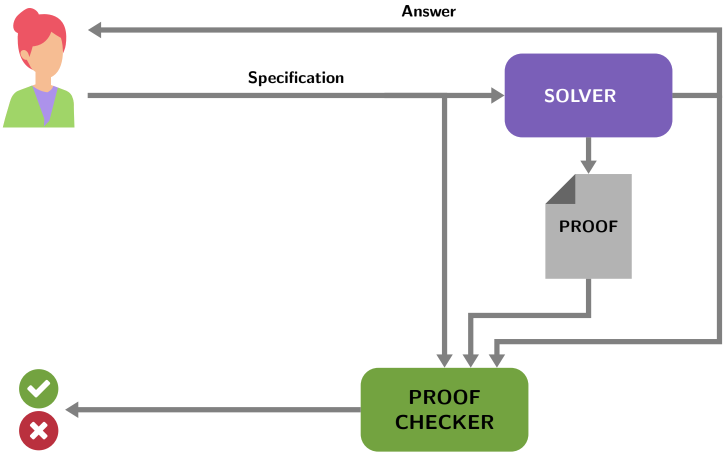

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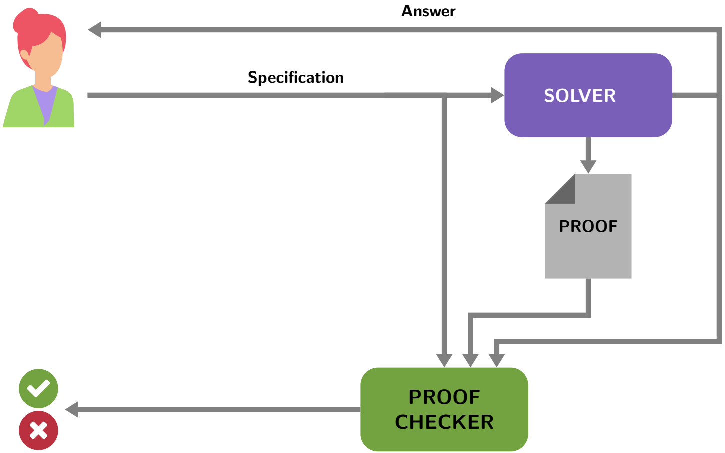

Software and Hardware Bugs

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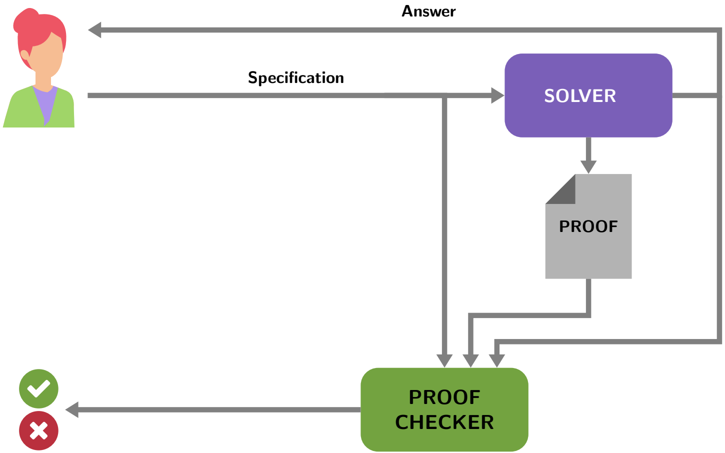

Debugging Support

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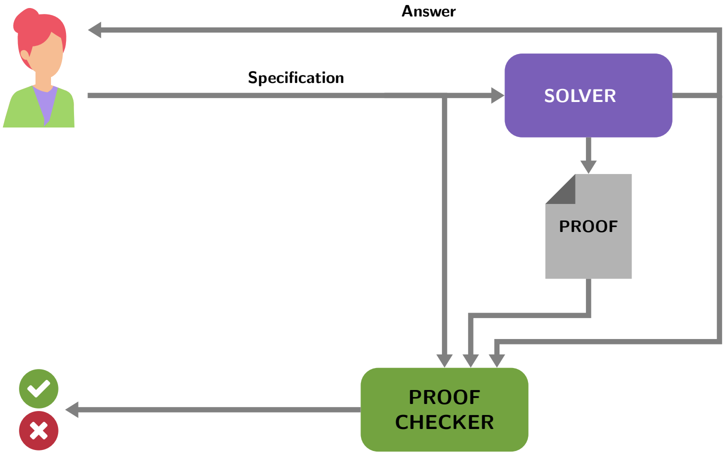

Auditable Record

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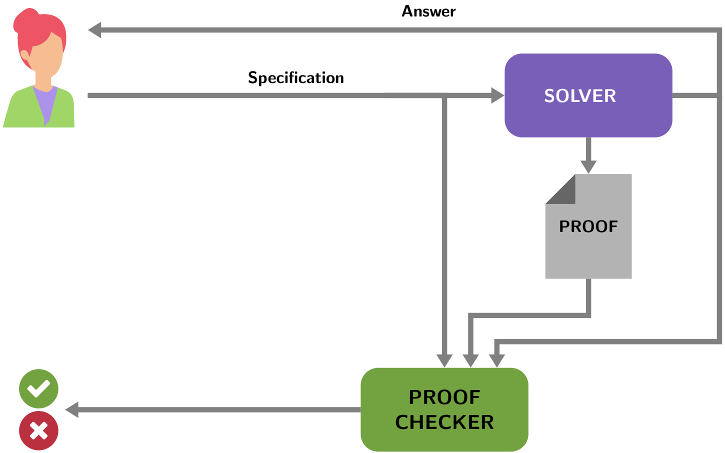

Performance Analysis

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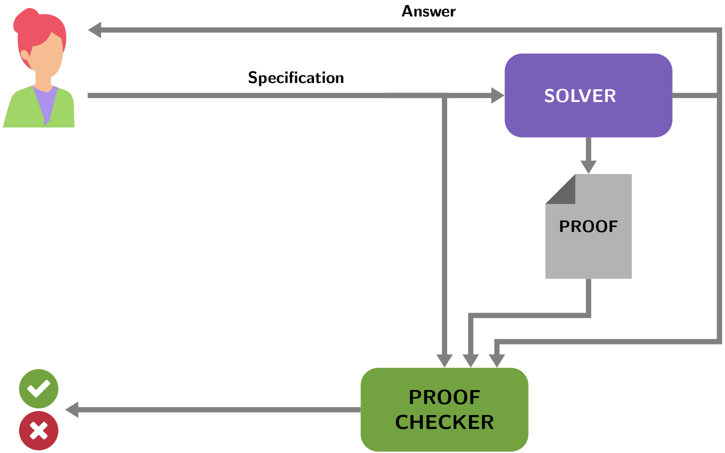
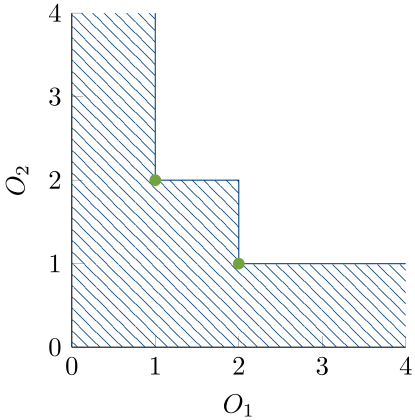


Paradigm	Proof format
SAT	DRAT
MaxSAT/PBO	VeriPB
SMT	Various formats
K Compilation	CPOG
...	...
MO-MaxSAT	This talk

THE P-Minimal ALGORITHM

MULTI-OBJECTIVE SOLUTION-IMPROVING SEARCH

Init (Solver $\leftarrow F$)



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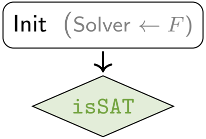

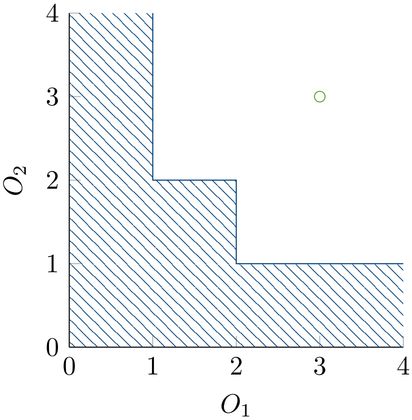

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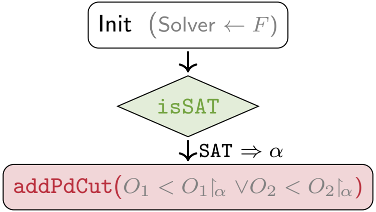

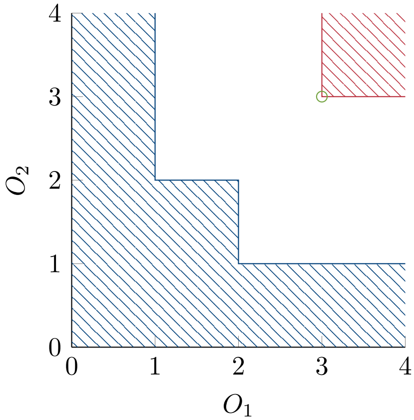

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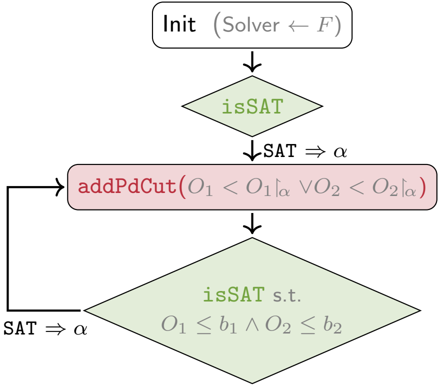

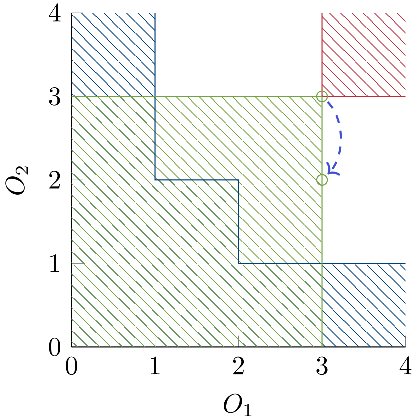

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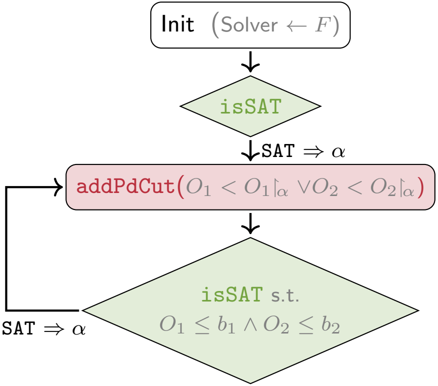

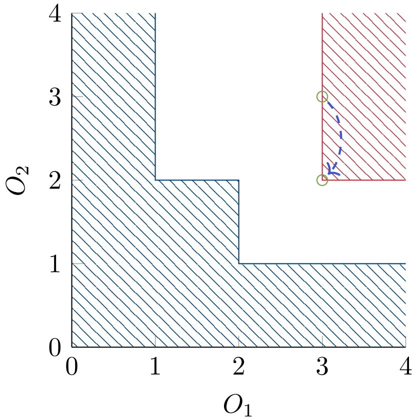

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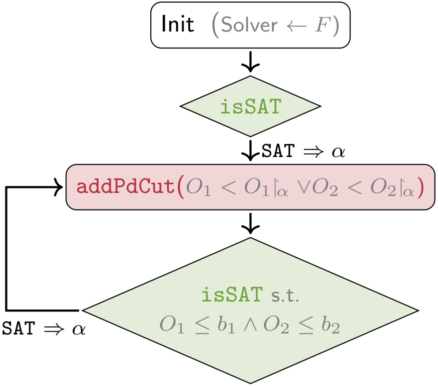

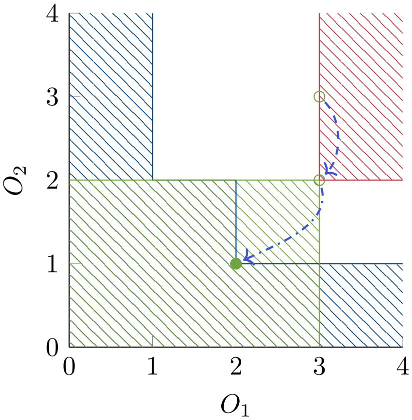

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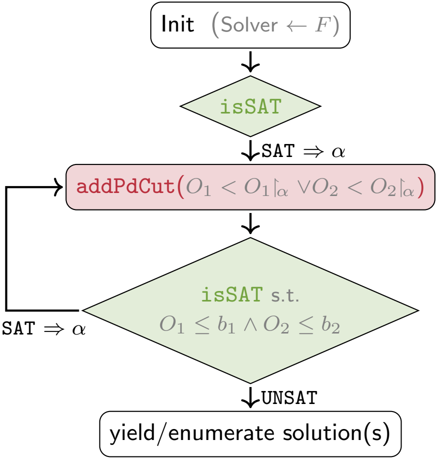

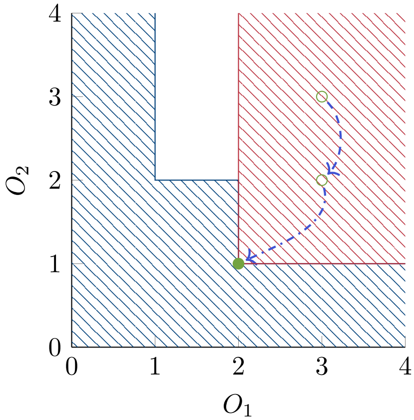

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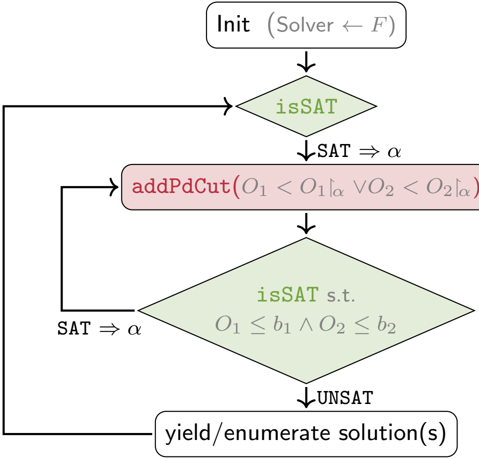

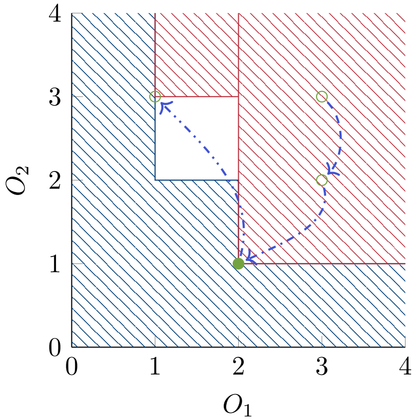

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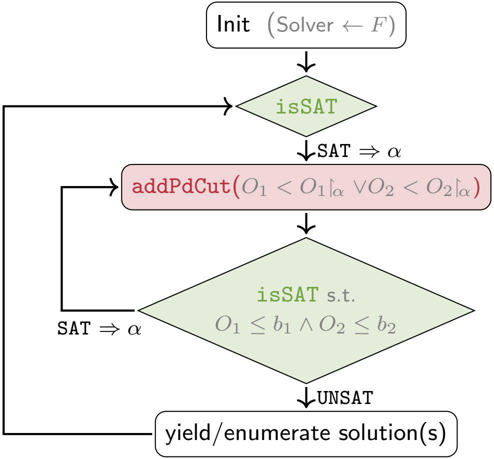

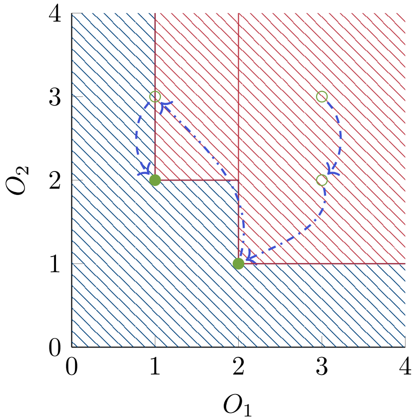

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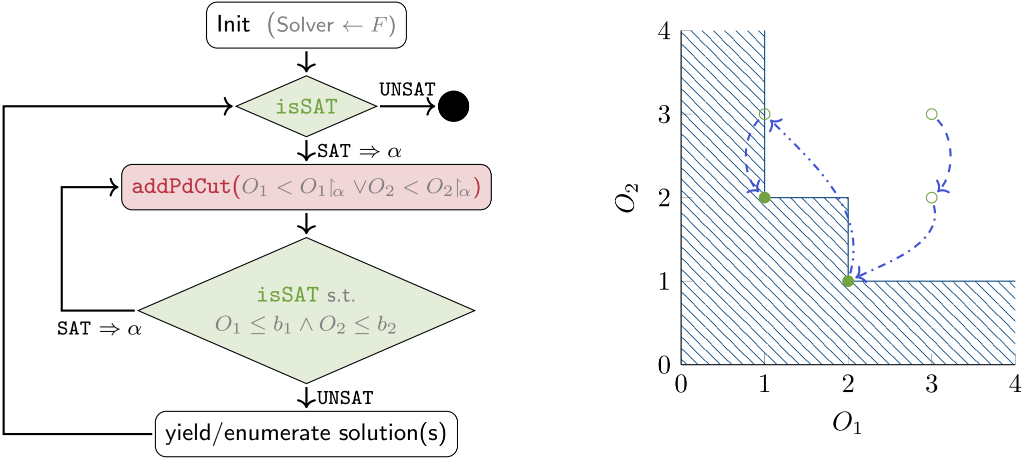

The VeriPB Proof System [BGMN23]

Pseudo-Boolean Cutting Planes Proofs

- Pseudo-Boolean Constraints

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PSEUDO-BOOLEAN CUTTING PLANES PROOFS

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- Recently: loading pre-order $\preceq$ for expressing preference (designed for symmetry breaking)
  - When patching up α to α', should show that $\alpha' \preceq \alpha$

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STRATEGY FOR GETTING VERIPB-BASED PROOFS

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   - SAT solver reasoning
   - PB-to-CNF encodings
   - MO-specific reasoning steps

PARETO DOMINANCE IN VERIPB PROOFS

Given $O_1, \ldots, O_p$
Required VeriPB order:
formula (over two copies of variables) that is true
iff α (weakly) dominates β ($\alpha \preceq \beta$)

PARETO DOMINANCE IN VERIPB PROOFS

USING VERIPB FOR MO PROBLEMS

Given $O_1, \ldots, O_p$
Required VERIPB order:
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VERIPB Pareto order

$$O_i\restriction_\alpha \leq O_i\restriction_\beta, \quad \text{for } i = 1, \ldots, p$$

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Syntactic restrictions

- First step in proof must load the Pareto order
- Order must never be changed

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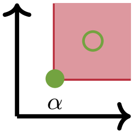
Guarantee

For each non-dominated point at least one solution explicitly appears in the proof

CERTIFYING PARETO DOMINANCE CUTS

THE BUILDING BLOCK FOR ALL ALGORITHMS



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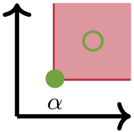

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1. Reified objective CNF encoding

$$w_1 \Leftrightarrow O_1 < O_1\restriction_\alpha$$
$$w_2 \Leftrightarrow O_2 < O_2\restriction_\alpha$$

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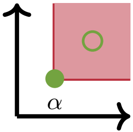


$$O_1 < O_1{\restriction_\alpha} \vee O_2 < O_2{\restriction_\alpha}$$

1. Reified objective CNF encoding

$$w_1 \Leftrightarrow O_1 < O_1{\restriction_\alpha}$$

$$w_2 \Leftrightarrow O_2 < O_2{\restriction_\alpha}$$

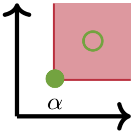
2. Map each weakly dominated solution to α

   (Redundant with α as witness)

$$\overline{w}_1 \wedge \overline{w}_2 \Rightarrow \alpha$$

CERTIFYING PARETO DOMINANCE CUTS

THE BUILDING BLOCK FOR ALL ALGORITHMS



$$O_1 < O_1\restriction_\alpha \vee O_2 < O_2\restriction_\alpha$$

1. Reified objective CNF encoding

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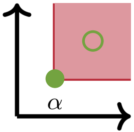

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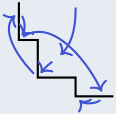
PROOF LOGGING MO-MAXSAT ALGORITHMS

PUTTING EVERYTHING TOGETHER

P-Minimal

[SBTL17]



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- PD cuts

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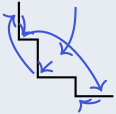

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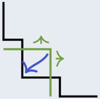
Lower-Bounding

[CLM23]



Upper-bounds irrelevant
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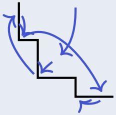

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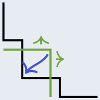

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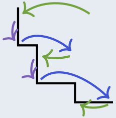
BiOptSat

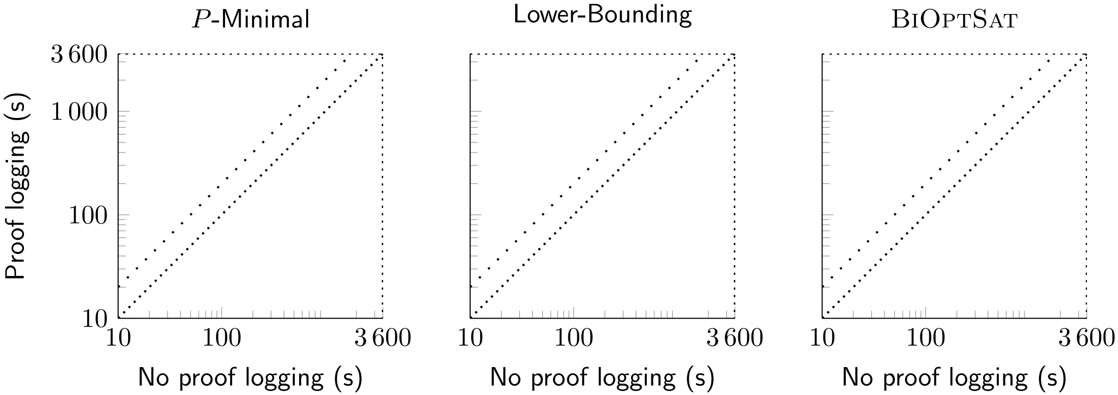
[JBNJ24]



- Derive lower-bound on first objective
- Certify PD cut
- Strengthen PD cut based on known lower-bound

PROOF LOGGING OVERHEAD

HOW EXPENSIVE IS THIS



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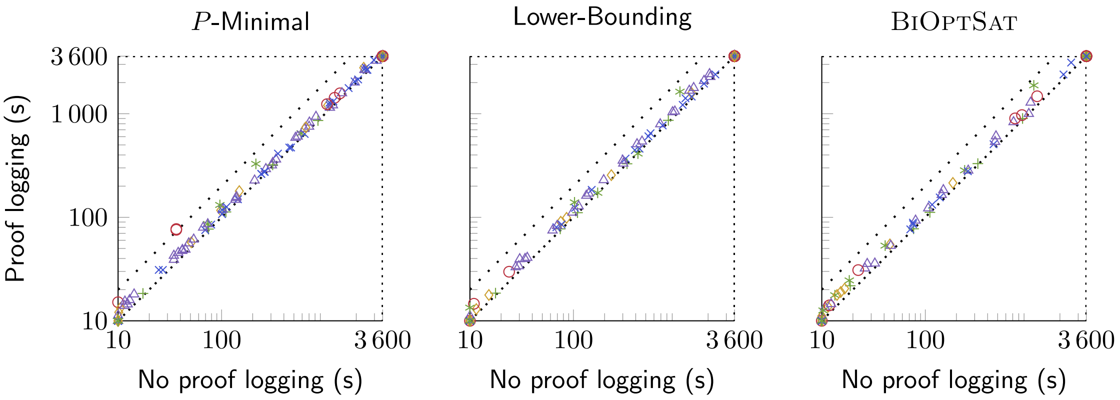

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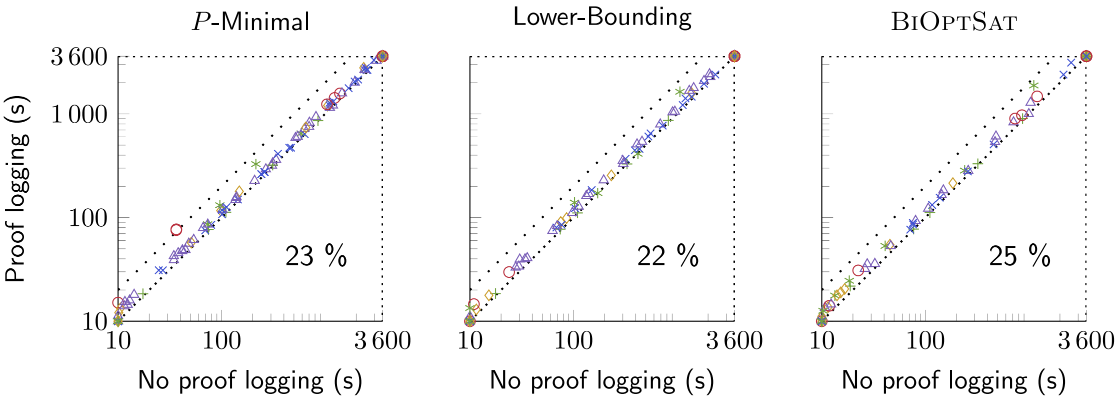

OUTLINE

1. Problem Setting
2. Background
3. Proofs for MO-MaxSAT
4. Conclusions

LIMITATIONS & FUTURE WORK

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  Can build on orders with auxiliary variables (see talk Markus)

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- Efficient checking (pboxide?)

SUMMARY AND CONCLUSIONS

- MO-MaxSAT certificates that all non-dominated points were discovered
- Includes techniques such as core-boosting & PB-to-CNF encodings
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Paper, slides, code, and contact:

`christophjabs.info/tacas25`

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